Programme 12

Matrix algebra

Frames 1 to 112

Learning outcomes

When you have completed this Programme you will be able to:

- Determine whether a matrix is singular or non-singular
- Determine the rank of a matrix
- Determine the consistency of a set of linear equations and hence demonstrate the uniqueness of their solution
- Obtain the solution of a set of simultaneous linear equations by using matrix inversion, by row transformation, by Gaussian elimination and by triangular decomposition
- Obtain the eigenvalues and corresponding eigenvectors of a square matrix
- Demonstrate the validity of the Cayley–Hamilton theorem
- Solve systems of first-order ordinary differential equations using eigenvalue and eigenvector methods
- Construct the modal matrix from the eigenvectors of a matrix and the spectral matrix from the eigenvalues
- Solve systems of second-order ordinary differential equations using diagonalisation
- Use matrices to represent transformations between coordinate systems

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 4 Determinants and 5 Matrices

Singular and non-singular matrices

1

Every square matrix **A** has associated with it a number called the determinant of **A** and denoted by $|\mathbf{A}|$. If $|\mathbf{A}| \neq 0$ then **A** is called a *non-singular* matrix. Otherwise if $|\mathbf{A}| = 0$, then **A** is called a *singular* matrix.

Example 1

Is $\mathbf{A} = \begin{pmatrix} 1 & 2 & 8 \\ 4 & 7 & 6 \\ 9 & 5 & 3 \end{pmatrix}$ singular or non-singular?

$$\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} 1 & 2 & 8 \\ 4 & 7 & 6 \\ 9 & 5 & 3 \end{vmatrix} \\
&= 1\begin{vmatrix} 7 & 6 \\ 5 & 3 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 9 & 3 \end{vmatrix} + 8\begin{vmatrix} 4 & 7 \\ 9 & 5 \end{vmatrix} \\
&= (21 - 30) - 2(12 - 54) + 8(20 - 63) \\
&= -9 + 84 - 344 \\
&= -269
\end{aligned}$$

Because $|\mathbf{A}| \neq 0$ then **A** is non-singular.

Example 2

Is $\mathbf{A} = \begin{pmatrix} 3 & 9 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 4 \end{pmatrix}$ singular or non-singular?

A is ....

2

singular

Because

$$\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} 3 & 9 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 4 \end{vmatrix} \\
&= 3(20 - 42) - 9(4 - 12) + 2(7 - 10) \\
&= -66 + 72 - 6 \\
&= 0
\end{aligned}$$

Because $|\mathbf{A}| = 0$ then $|\mathbf{A}|$ is singular.

Exercise

Determine whether each of the following is singular or non-singular.

1 $|\mathbf{A}| = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$ **2** $|\mathbf{B}| = \begin{pmatrix} 3 & -4 \\ -6 & 8 \end{pmatrix}$

3 $|\mathbf{C}| = \begin{pmatrix} 4 & 1 & -2 \\ 1 & 7 & 3 \\ 5 & 8 & 1 \end{pmatrix}$ **4** $|\mathbf{D}| = \begin{pmatrix} 3 & 2 & 4 \\ 5 & 1 & 6 \\ 2 & 0 & 3 \end{pmatrix}$

3

1 non-singular	**2** singular
3 singular	**4** non-singular

Because

Straightforward evaluation of the relevant determinants gives

1 $|\mathbf{A}| = 2$ **2** $|\mathbf{B}| = 0$

3 $|\mathbf{C}| = 0$ **4** $|\mathbf{D}| = -5$

Closely related to the notion of the singularity or otherwise of a square matrix is the notion of **rank** of a general $n \times m$ matrix.

Rank of a matrix

The rank of an $n \times m$ matrix $\mathbf{A}$ is the order of the largest square, non-singular sub-matrix. That is, the largest square sub-matrix whose determinant is non-zero. If $n = m$, so making $\mathbf{A}$ itself square, then this sub-matrix could be the matrix $\mathbf{A}$ itself.

Example

To find the rank of the matrix $\mathbf{A} = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ we note that

$$|\mathbf{A}| = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \ldots\ldots\ldots\ldots$$

4

$\boxed{0}$

Because

$$|\mathbf{A}| = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

$$= 3(12-15) - 4(6-12) + 5(5-8)$$

$$= -9 + 24 - 15 = 0$$

Therefore we can say that the rank of **A** is ....

5

$\boxed{\text{not } 3}$

Because

$|\mathbf{A}| = 0$ and therefore **A** is singular.

Now try a sub-matrix of order 2.

$\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = -2 \neq 0$. Therefore the rank of **A** is ....

6

$\boxed{2}$

Because

The largest square, non-singular sub-matrix of **A** has order 2 therefore **A** has rank 2.

This method of finding the rank of a matrix can be a very hit and miss affair and a better, more systematic method is to use **elementary operations** and the notion of an **equivalent matrix**.

Next frame

Elementary operations and equivalent matrices

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Each of the following row operations on matrix **A** produces a *row equivalent matrix* **B**, where the order and rank of **B** is the same as that of **A**. We write $\mathbf{A} \sim \mathbf{B}$.

1 Interchanging two rows
2 Multiplying each element of a row by the same non-zero scalar quantity
3 Adding or subtracting corresponding elements from those of another row

are operations called *elementary row operations*. There is a corresponding set of three *elementary column operations* that can be used to form *column equivalent matrices*.

Example 1

Given $\mathbf{A} = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then

$$\begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \sim \begin{pmatrix} 0 & -2 & -4 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$ by subtracting 3 times each element of row 2 from row 1

$$\sim \begin{pmatrix} 0 & -2 & -4 \\ 1 & 2 & 3 \\ 0 & -3 & -6 \end{pmatrix}$$ by subtracting 4 times each element of row 2 from row 3

$$\sim \begin{pmatrix} 0 & -3 & -6 \\ 1 & 2 & 3 \\ 0 & -3 & -6 \end{pmatrix}$$ by multiplying each element of row 1 by 3/2

$$\sim \begin{pmatrix} 0 & -3 & -6 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$ by subtracting corresponding elements of row 1 from row 3

$$= \mathbf{B}$$

The row of zeros in matrix **B** means that its determinant is zero and so its rank is not 3. The largest sub-matrix with non-zero determinant has order 2 and so the rank of **B** is 2. Because matrix **B** is row equivalent to matrix **A** we can say that the rank of **A** is also 2.

Example 2

Determine the rank of $\mathbf{A} = \begin{pmatrix} 1 & 2 & 8 \\ 4 & 7 & 6 \\ 9 & 5 & 3 \end{pmatrix}$

By taking 4 times the elements of row 1 from row 2 we obtain the equivalent matrix ....

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$$\boxed{\begin{pmatrix} 1 & 2 & 8 \\ 0 & -1 & -26 \\ 9 & 5 & 3 \end{pmatrix}}$$

By taking 9 times the elements of row 1 from row 3 we obtain the equivalent matrix ....

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$$\boxed{\begin{pmatrix} 1 & 2 & 8 \\ 0 & -1 & -26 \\ 0 & -13 & -69 \end{pmatrix}}$$

By multiplying the elements of row 2 by -13 we obtain the equivalent matrix ....

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$$\boxed{\begin{pmatrix} 1 & 2 & 8 \\ 0 & 13 & 338 \\ 0 & -13 & -69 \end{pmatrix}}$$

By adding corresponding elements of row 2 to row 3 we obtain the equivalent matrix ....

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$$\boxed{\begin{pmatrix} 1 & 2 & 8 \\ 0 & 13 & 269 \\ 0 & 0 & -69 \end{pmatrix}}$$

Because all the elements below the main diagonal of this matrix are zero we call the matrix an *upper triangular matrix*. By inspection we can see that the determinant of this triangular matrix is non-zero, being the product of its three diagonal elements $1 \times 13 \times (-69) = -897$. Therefore its rank is 3 and so the rank of matrix **A** is also 3.

Try another one for yourself.

Example 3

The rank of $\mathbf{A} = \begin{pmatrix} 3 & 9 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 4 \end{pmatrix}$ is ....

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$$\boxed{2}$$

Because

$$\mathbf{A} = \begin{pmatrix} 3 & 9 & 2 \\ 1 & 5 & 6 \\ 2 & 7 & 4 \end{pmatrix} \sim \begin{pmatrix} 0 & -6 & -16 \\ 1 & 5 & 6 \\ 2 & 7 & 4 \end{pmatrix} \quad \text{Subtracting 3 times row 2 from row 1}$$

$$\sim \begin{pmatrix} 0 & -6 & -16 \\ 1 & 5 & 6 \\ 0 & -3 & -8 \end{pmatrix} \quad \text{Subtracting 2 times row 2 from row 3}$$

$$\sim \begin{pmatrix} 0 & 3 & 8 \\ 1 & 5 & 6 \\ 0 & -3 & -8 \end{pmatrix} \quad \text{Multiplying row 1 by } -1/2$$

$$\sim \begin{pmatrix} 0 & 3 & 8 \\ 1 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Adding row 1 to row 3}$$

$$\sim \begin{pmatrix} 1 & 5 & 6 \\ 0 & 3 & 8 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Interchanging rows 1 and 2}$$

▶

and $\begin{vmatrix} 1 & 5 & 6 \\ 0 & 3 & 8 \\ 0 & 0 & 0 \end{vmatrix} = 0$. So the rank of this matrix is not 3. The largest square sub-matrix of this matrix with non-zero determinant is, by inspection, of order 2 and so the rank of this matrix, and hence the rank of the equivalent matrix **A** is 2.

Finally try a non-square matrix.

Example 4

The rank of $\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 & 1 \\ 0 & 8 & 2 & 4 \\ 1 & 7 & 3 & 2 \end{pmatrix}$ is ....

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$\boxed{3}$

Because

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 & 1 \\ 0 & 8 & 2 & 4 \\ 1 & 7 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & -12 & -3 & -3 \\ 0 & 8 & 2 & 4 \\ 1 & 7 & 3 & 2 \end{pmatrix}$$ Subtracting 2 times row 3 from row 1

$$\sim \begin{pmatrix} 0 & -8 & -2 & -2 \\ 0 & 8 & 2 & 4 \\ 1 & 7 & 3 & 2 \end{pmatrix}$$ Multiplying row 1 by 2/3

$$\sim \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 8 & 2 & 4 \\ 1 & 7 & 3 & 2 \end{pmatrix}$$ Adding row 2 to row 1

It is possible to find a 3×3 sub-matrix of this matrix that has non-zero determinant, namely

$$\begin{pmatrix} 0 & 0 & 2 \\ 8 & 2 & 4 \\ 7 & 3 & 2 \end{pmatrix} \text{ where } \begin{vmatrix} 0 & 0 & 2 \\ 8 & 2 & 4 \\ 7 & 3 & 2 \end{vmatrix} = 2(24 - 14) = 20.$$

Consequently, this matrix and hence matrix **A** has rank 3.

Consistency of a set of equations

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In solving sets of simultaneous equations, we can express the equations in matrix form. For example

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

can be written in the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

i.e. $$\mathbf{Ax} = \mathbf{b}$$

The set of three equations is said to be *consistent* if solutions for x_1, x_2, x_3 exist and *inconsistent* if no such solutions can be found.

In practice, we can solve the equations by operating on the *augmented coefficient matrix*, i.e. we write the constant terms as a fourth column of the coefficient matrix to form $\mathbf{A}_\mathrm{b}$.

$$\mathbf{A}_\mathrm{b} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}$$

which, of course, is a (3×4) matrix.

The general test for consistency is then:

A set of n simultaneous equations in n unknowns is consistent if the rank of the coefficient matrix $\mathbf{A}$ is equal to the rank of the augmented matrix $\mathbf{A}_\mathrm{b}$.

If the rank of $\mathbf{A}$ is less than the rank of $\mathbf{A}_\mathrm{b}$, then the equations are inconsistent and have no solution.

Make a note of this test. It can save time in working

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Example

If $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ then

$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ and $\mathbf{A}_\mathrm{b} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 5 \end{pmatrix}$

Rank of $\mathbf{A}$: $\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0 \quad \therefore$ rank of $\mathbf{A} = 1$

Rank of $\mathbf{A}_\mathrm{b}$: $\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0$ as before

but $\begin{vmatrix} 3 & 4 \\ 6 & 5 \end{vmatrix} = 15 - 24 = -9 \quad \therefore$ rank of $\mathbf{A}_\mathrm{b} = 2$

In this case, rank of $\mathbf{A}$ < rank of $\mathbf{A}_\mathrm{b}$

$\therefore$ ....

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$\boxed{\text{no solution exists}}$

Remember that, for consistency,

rank of $\mathbf{A} =$ ....

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$\boxed{\text{rank of } \mathbf{A}_b}$

Uniqueness of solutions

1 With a set of n equations in n unknowns, the equations are consistent if the coefficient matrix $\mathbf{A}$ and the augmented matrix $\mathbf{A}_b$ are each of rank n. There is then a *unique* solution for the n equations.

Note that if the rank of $\mathbf{A} = n$ then $\mathbf{A}$ is a non-singular sub-matrix of $\mathbf{A}_b$ and so the rank of $\mathbf{A}_b = n$ also. Therefore there is no need to test for the rank of $\mathbf{A}_b$ in this case.

2 If the rank of $\mathbf{A}$ and that of $\mathbf{A}_b$ is m, where $m < n$, then the matrix $\mathbf{A}$ is singular, i.e. $|\mathbf{A}| = \mathbf{0}$, and there will be an *infinite number* of solutions for the equations.

3 As we have already seen, if the rank of $\mathbf{A} <$ the rank of $\mathbf{A}_b$, then *no solution* exists.

Copy these up in your record book; they are important

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Writing the results in a slightly different way:

With a set of n equations in n unknowns, checking the rank of the coefficient matrix $\mathbf{A}$ and that of the augmented matrix $\mathbf{A}_b$ enables us to see whether

(a) a unique solution exists

rank $\mathbf{A} =$ rank $\mathbf{A}_b = n$

(b) an infinite number of solutions exist

rank $\mathbf{A} =$ rank $\mathbf{A}_b = m < n$

(c) no solution exists

rank $\mathbf{A} <$ rank $\mathbf{A}_b$

Example

$$\begin{pmatrix} -4 & 5 \\ -8 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$$

Finding the rank of $\mathbf{A}$ and of $\mathbf{A}_b$ leads us to the conclusion that

....

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there is an infinite number of solutions

Because

$$\mathbf{A} = \begin{pmatrix} -4 & 5 \\ -8 & 10 \end{pmatrix} \text{ and } \mathbf{A}_b = \begin{pmatrix} -4 & 5 & -3 \\ -8 & 10 & -6 \end{pmatrix}$$

Rank of $\mathbf{A}$: $\begin{vmatrix} -4 & 5 \\ -8 & 10 \end{vmatrix} = -40 + 40 = 0 \quad \therefore$ Rank of $\mathbf{A} = 1$

Rank of $\mathbf{A}_b$: $\begin{vmatrix} -4 & 5 \\ -8 & 10 \end{vmatrix} = 0; \begin{vmatrix} 5 & -3 \\ 10 & -6 \end{vmatrix} = 0; \begin{vmatrix} -4 & -3 \\ -8 & -6 \end{vmatrix} = 0$

$\therefore$ Rank of $\mathbf{A}_b = 1$

$\therefore$ Rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = 1$

But there are two equations in two unknowns, i.e. $n = 2$

$\therefore$ Rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = 1 < n$

$\therefore$ Infinite number of solutions.

The solutions can be written as x_1 arbitrary and $x_2 = \dfrac{4x_1 - 3}{5}$.

You will recall that, for a unique solution of n equations in n unknowns

....

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rank $\mathbf{A}$ = rank $\mathbf{A}_b = n$

Now for some examples for you to try. In each of the following cases, apply the rank tests to determine the nature of the solutions. Do not solve the sets of equations.

Example 1

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix} \text{ and } \mathbf{A}_b = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 3 & 4 & 2 & -2 \\ 1 & 4 & 3 & 3 \end{pmatrix}$$

Finish it off and we find that ....

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a unique solution exists

Because

$n = 3$; rank of $\mathbf{A} = 3$; rank of $\mathbf{A}_b = 3$.

$\therefore$ rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = 3 = n$ $\therefore$ Solution unique

And this one.

Example 2

$$\begin{pmatrix} 2 & -1 & 7 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

This time we find that ....

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no solution is possible

Because

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 7 \\ 4 & 2 & 2 \\ 3 & 1 & 3 \end{pmatrix}; \quad \mathbf{A}_b = \begin{pmatrix} 2 & -1 & 7 & 2 \\ 4 & 2 & 2 & 5 \\ 3 & 1 & 3 & 1 \end{pmatrix}$$

$n = 3$; rank of $\mathbf{A} = 2$; rank of $\mathbf{A}_b = 3$

$\therefore$ rank of $\mathbf{A}$ < rank of $\mathbf{A}_b$

$\therefore$ No solution exists

and finally

Example 3

$$\begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 4 \\ 2 & 5 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

In this case, we find that ....

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infinite number of solutions possible

Because

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 4 \\ 2 & 5 & 1 \end{pmatrix} \text{ and } \mathbf{A}_b = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 5 & 1 & 3 \end{pmatrix}$$

Rank of **A**:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 3 & 4 \\ 2 & 5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 7 \\ 2 & 5 & 1 \end{pmatrix} \quad \text{Subtracting row 1 from row 2}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 7 \\ 0 & 1 & 7 \end{pmatrix} \quad \text{Subtracting 2 times row 1 from row 2}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Subtracting row 2 from row 3}$$

and so rank of **A** is 2 by inspection.

Rank of $\mathbf{A}_b$:

$$\mathbf{A}_b = \begin{pmatrix} 1 & 2 & -3 & 1 \\ 1 & 3 & 4 & 2 \\ 2 & 5 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 7 & 1 \\ 2 & 5 & 1 & 3 \end{pmatrix} \quad \text{Subtracting row 1 from row 2}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 7 & 1 \\ 0 & 1 & 7 & 1 \end{pmatrix} \quad \text{Subtracting 2 times row 1 from row 2}$$

$$\sim \begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Subtracting row 2 from row 3}$$

and so rank of $\mathbf{A}_b$ is 2 by inspection.

Therefore rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = 2 < n$ (that is 3), therefore there is an infinite number of solutions.

Now let us move on to a new section of the work

Solution of sets of equations

1 *Inverse method*

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Let us work through an example by way of explanation.

Example 1

To solve
$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= 4 \\ 2x_1 - x_2 + 2x_3 &= 10 \\ x_1 - 3x_2 - 4x_3 &= 5. \end{aligned}$$

We first write this in matrix form, which is ....

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$$\boxed{\begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}}$$

Then if $\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{pmatrix}$ then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}$

where $\mathbf{A}^{-1}$ is the *inverse* of $\mathbf{A}$.

To find $\mathbf{A}^{-1}$

(a) Form the determinant of $\mathbf{A}$ and evaluate it.

$$|\mathbf{A}| = \begin{vmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{vmatrix} = 3(4+6) - 2(-8-2) - 1(-6+1) = 55$$

(b) Form a new matrix $\mathbf{C}$ consisting of the cofactors of the elements in $\mathbf{A}$.

The cofactor of any one element is its minor together with its 'place sign'

i.e. $\mathbf{C} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$

where A_{11} is the cofactor of a_{11} in $\mathbf{A}$.

$$A_{11} = \begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} = 10; \quad A_{12} = -\begin{vmatrix} 2 & 2 \\ 1 & -4 \end{vmatrix} = 10;$$

$$A_{13} = \begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} = -5$$

$$A_{21} = -\begin{vmatrix} 2 & -1 \\ -3 & -4 \end{vmatrix} = 11; \quad A_{22} = \begin{vmatrix} 3 & -1 \\ 1 & -4 \end{vmatrix} = -11;$$

$$A_{23} = -\begin{vmatrix} 3 & 2 \\ 1 & -3 \end{vmatrix} = 11$$

$A_{31} =$ ....; $\quad A_{32} =$ ....; $\quad A_{33} =$ ....

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$$A_{31} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3; \quad A_{32} = -\begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix} = -8; \quad A_{33} = \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} = -7$$

$$\text{So } \mathbf{C} = \begin{pmatrix} 10 & 10 & -5 \\ 11 & -11 & 11 \\ 3 & -8 & -7 \end{pmatrix}$$

We now write the transpose of $\mathbf{C}$, i.e. $\mathbf{C}^{\mathrm{T}}$ in which we write rows as columns and columns as rows.

$$\mathbf{C}^{\mathrm{T}} = \ldots\ldots\ldots\ldots$$

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$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}$$

This is called the *adjoint* (adj) of the original matrix $\mathbf{A}$

i.e. $\operatorname{adj} \mathbf{A} = \mathbf{C}^{\mathrm{T}}$

Then the inverse of $\mathbf{A}$, i.e. $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \times \mathbf{C}^{\mathrm{T}} = \frac{1}{55}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}$$

As a check that all the calculations have been done correctly and without error, the product of matrix $\mathbf{A}$ with its adjoint should be equal to the unit matrix multiplied by the determinant of $\mathbf{A}$. That is

$$\mathbf{A} \times \operatorname{adj} \mathbf{A} = \det \mathbf{A} \times \mathbf{I}$$

For this case

$$\begin{aligned} \mathbf{A} \times \operatorname{adj} \mathbf{A} &= \begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{pmatrix}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix} \\ &= \begin{pmatrix} 55 & 0 & 0 \\ 0 & 55 & 0 \\ 0 & 0 & 55 \end{pmatrix} \\ &= \det \mathbf{A} \times \mathbf{I} \end{aligned}$$

Thus all is well. We can now continue to find the solution.

$$\text{So } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix} \text{ becomes}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{55}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix} = \ldots\ldots\ldots\ldots$$

$$\boxed{x_1 = 3; \quad x_2 = -2; \quad x_3 = 1}$$

Because

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{55}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}$$

$$= \frac{1}{55}\begin{pmatrix} 40 & +110 & +15 \\ 40 & -110 & -40 \\ -20 & +110 & -35 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$\therefore\ x_1 = 3; \quad x_2 = -2; \quad x_3 = 1$

The method is the same every time.

To solve $\mathbf{Ax} = \mathbf{b} \qquad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

To find $\mathbf{A}^{-1}$

(1) Evaluate $|\mathbf{A}|$

If $|\mathbf{A}| \neq 0$ then proceed to (2)

If $|\mathbf{A}| = 0$ then there is no inverse and hence no unique solution. Later we shall discover how to determine whether there is an infinity of solutions or none.

(2) Form $\mathbf{C}$, the matrix of cofactors of $\mathbf{A}$

(3) Write $\mathbf{C}^{\mathrm{T}}$, the transpose of $\mathbf{C}$

(4) Then $\mathbf{A}^{-1} = \dfrac{1}{|\mathbf{A}|} \times \mathbf{C}^{\mathrm{T}}$.

Now apply the method to Example 2.

Example 2

$$\begin{aligned} 4x_1 + 5x_2 + \ \ x_3 &= 2 \\ x_1 - 2x_2 - 3x_3 &= 7 \\ 3x_1 - \ \ x_2 - 2x_3 &= 1. \end{aligned}$$

$x_1 = \ldots\ldots\ldots\ldots; \qquad x_2 = \ldots\ldots\ldots\ldots; \qquad x_3 = \ldots\ldots\ldots\ldots$

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$$\boxed{x_1 = -2; \quad x_2 = 3; \quad x_3 = -5}$$

Here is the complete working.

$$\mathbf{A} = \begin{pmatrix} 4 & 5 & 1 \\ 1 & -2 & -3 \\ 3 & -1 & -2 \end{pmatrix} \therefore |\mathbf{A}| = \begin{vmatrix} 4 & 5 & 1 \\ 1 & -2 & -3 \\ 3 & -1 & -2 \end{vmatrix} = -26$$

$$\mathbf{C} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$A_{11} = \begin{vmatrix} -2 & -3 \\ -1 & -2 \end{vmatrix} = 1 \qquad A_{12} = -\begin{vmatrix} 1 & -3 \\ 3 & -2 \end{vmatrix} = -7 \quad A_{13} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = 5$$

$$A_{21} = -\begin{vmatrix} 5 & 1 \\ -1 & -2 \end{vmatrix} = 9 \qquad A_{22} = \begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = -11 \quad A_{23} = -\begin{vmatrix} 4 & 5 \\ 3 & -1 \end{vmatrix} = 19$$

$$A_{31} = \begin{vmatrix} 5 & 1 \\ -2 & -3 \end{vmatrix} = -13 \quad A_{32} = -\begin{vmatrix} 4 & 1 \\ 1 & -3 \end{vmatrix} = 13 \quad A_{33} = \begin{vmatrix} 4 & 5 \\ 1 & -2 \end{vmatrix} = -13$$

$$\therefore \mathbf{C} = \begin{pmatrix} 1 & -7 & 5 \\ 9 & -11 & 19 \\ -13 & 13 & -13 \end{pmatrix} \quad \therefore \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 1 & 9 & -13 \\ -7 & -11 & 13 \\ 5 & 19 & -13 \end{pmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \times \mathbf{C}^{\mathrm{T}} = -\frac{1}{26}\begin{pmatrix} 1 & 9 & -13 \\ -7 & -11 & 13 \\ 5 & 19 & -13 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix} = -\frac{1}{26}\begin{pmatrix} 1 & 9 & -13 \\ -7 & -11 & 13 \\ 5 & 19 & -13 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$$

$$= -\frac{1}{26}\begin{pmatrix} 2 & +63 & -13 \\ -14 & -77 & +13 \\ 10 & +133 & -13 \end{pmatrix}$$

$$= -\frac{1}{26}\begin{pmatrix} 52 \\ -78 \\ 130 \end{pmatrix} = -\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

$\therefore \; x_1 = -2; \quad x_2 = 3; \quad x_3 = -5$

With a set of four equations with four unknowns, the method becomes somewhat tedious as there are then sixteen cofactors to be evaluated and each one is a third-order determinant! There are, however, other methods that can be applied – so let us see method 2.

2 Row transformation method

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Elementary row transformations that can be applied are as follows

(a) Interchange any two rows.
(b) Multiply (or divide) every element in a row by a non-zero scalar (constant) k.
(c) Add to (or subtract from) all the elements of any row k times the corresponding elements of any other row.

Equivalent matrices
Two matrices, **A** and **B**, are said to be equivalent if **B** can be obtained from **A** by a sequence of elementary transformations.

Solutions of equations
The method is best described by working through a typical example.

Example 1

Solve
$$\begin{aligned} 2x_1 + x_2 + x_3 &= 5 \\ x_1 + 3x_2 + 2x_3 &= 1 \\ 3x_1 - 2x_2 - 4x_3 &= -4. \end{aligned}$$

This can be written $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$

and for convenience we introduce the unit matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$$

where $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ may be regarded as the coefficient of $\begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$

We then form the combined coefficient matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 3 & -2 & -4 & 0 & 0 & 1 \end{pmatrix}$$

and work on this matrix from now on.

On then to the next frame

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The rest of the working is mainly concerned with applying row transformations to convert the left-hand half of the matrix to a unit matrix and the right-hand side to the inverse, eventually obtaining

$$\begin{pmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{pmatrix}$$

with $a, b, c, \ldots g, h, i$ being evaluated in the process.

The following notation will be helpful to denote the transformation used:

(1) ~ (2) denotes 'interchange rows 1 and 2'
(3) − 2(1) denotes 'subtract twice row 1 from row 3', etc.

So off we go.

$$(1) \sim (2) \quad \begin{pmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & -2 & -4 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix}(2)-2(1)\\(3)-3(1)\end{matrix} \quad \begin{pmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & -5 & -3 & 1 & -2 & 0 \\ 0 & -11 & -10 & 0 & -3 & 1 \end{pmatrix}$$

$$(3)-2(2) \quad \begin{pmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & -5 & -3 & 1 & -2 & 0 \\ 0 & -1 & -4 & -2 & 1 & 1 \end{pmatrix}$$

$$-(2) \sim -(3) \quad \begin{pmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & -1 & -1 \\ 0 & 5 & 3 & -1 & 2 & 0 \end{pmatrix}$$

$$(3)-5(2) \quad \begin{pmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 4 & 2 & -1 & -1 \\ 0 & 0 & -17 & -11 & 7 & 5 \end{pmatrix}$$

$$\begin{matrix}(1)-3(2)\\(3)\div(-17)\end{matrix} \quad \begin{pmatrix} 1 & 0 & -10 & -6 & 4 & 3 \\ 0 & 1 & 4 & 2 & -1 & -1 \\ 0 & 0 & 1 & 11/17 & -7/17 & -5/17 \end{pmatrix}$$

$$\begin{matrix}(1)+10(3)\\(2)-4(3)\end{matrix} \quad \begin{pmatrix} 1 & 0 & 0 & 8/17 & -2/17 & 1/17 \\ 0 & 1 & 0 & -10/17 & 11/17 & 3/17 \\ 0 & 0 & 1 & 11/17 & -7/17 & -5/17 \end{pmatrix}$$

We now have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 8 & -2 & 1 \\ -10 & 11 & 3 \\ 11 & -7 & -5 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$$

$\therefore\ x_1 = \ldots\ldots\ldots\ldots;\quad x_2 = \ldots\ldots\ldots\ldots;\quad x_3 = \ldots\ldots\ldots\ldots$

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$$\boxed{x_1 = 2; \quad x_2 = -3; \quad x_3 = 4}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 40 & -2 & -4 \\ -50 & +11 & -12 \\ 55 & -7 & +20 \end{pmatrix} = \frac{1}{17}\begin{pmatrix} 34 \\ -51 \\ 68 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$x_1 = 2; \quad x_2 = -3; \quad x_3 = 4$

Of course, there is no set pattern of how to carry out the row transformations. It depends on one's ingenuity and every case is different. Here is a further example.

Example 2

$$\begin{aligned} 2x_1 - x_2 - 3x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 - 2x_2 - 5x_3 &= 2. \end{aligned}$$

First write the set of equations in matrix form – with the unit matrix included. This gives ....

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$$\boxed{\begin{pmatrix} 2 & -1 & -3 \\ 1 & 2 & 1 \\ 2 & -2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}$$

The combined coefficient matrix is now ....

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$$\boxed{\begin{pmatrix} 2 & -1 & -3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -2 & -5 & 0 & 0 & 1 \end{pmatrix}}$$

If we start off by interchanging the top two rows, we obtain a 1 at the beginning of the top row which is a help.

$$(1) \sim (2) \quad \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & -3 & 1 & 0 & 0 \\ 2 & -2 & -5 & 0 & 0 & 1 \end{pmatrix}$$

Now, if we subtract $2 \times$ row 1 from row 2
and $2 \times$ row 1 from row 3, we get

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$$\boxed{\begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -5 & -5 & 1 & -2 & 0 \\ 0 & -6 & -7 & 0 & -2 & 1 \end{pmatrix}}$$

Continuing with the same line of reasoning, we then have

$$\begin{array}{r} (2)-(3) \\ \\ \\ \end{array} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & -6 & -7 & 0 & -2 & 1 \end{pmatrix}$$

$$\begin{array}{r} (3)+6(2) \\ \\ \\ \end{array} \begin{pmatrix} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 5 & 6 & -2 & -5 \end{pmatrix}$$

$$\begin{array}{r} (1)-2(2) \\ (3)\div 5 \\ \\ \end{array} \begin{pmatrix} 1 & 0 & -3 & -2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{6}{5} & -\frac{2}{5} & -1 \end{pmatrix}$$

Notice the three diagonal 1s appearing at the left-hand end

What do you suggest we should do now?

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$$\boxed{\begin{array}{c}\text{Add three times row 3 to row 1} \\ \text{and subtract twice row 3 from row 2}\end{array}}$$

Right. That gives

$$\begin{array}{r} (1)+3(3) \\ (2)-3(3) \\ \\ \end{array} \begin{pmatrix} 1 & 0 & 0 & \frac{8}{5} & -\frac{1}{5} & -1 \\ 0 & 1 & 0 & -\frac{7}{5} & \frac{4}{5} & 1 \\ 0 & 0 & 1 & \frac{6}{5} & -\frac{2}{5} & -1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 8 & -1 & -5 \\ -7 & 4 & 5 \\ 6 & -2 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Now you can finish it off.

$x_1 = \ldots\ldots\ldots\ldots;\quad x_2 = \ldots\ldots\ldots\ldots;\quad x_3 = \ldots\ldots\ldots\ldots$

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$$\boxed{x_1 = -1;\quad x_2 = 3;\quad x_3 = -2}$$

Because

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 8 - \ \ 3 - 10 \\ -7 + 12 + 10 \\ 6 - \ \ 6 - 10 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -5 \\ 15 \\ -10 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

Let us now look at a somewhat similar method with rather fewer steps involved.

So move on

3 Gaussian elimination method

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Once again we will demonstrate the method by a typical example.

Example 1

$$\begin{aligned}2x_1 - 3x_2 + 2x_3 &= 9\\ 3x_1 + 2x_2 - x_3 &= 4\\ x_1 - 4x_2 + 2x_3 &= 6.\end{aligned}$$

We start off as usual

$$\begin{pmatrix}2 & -3 & 2\\ 3 & 2 & -1\\ 1 & -4 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}9\\ 4\\ 6\end{pmatrix}$$

We then form the *augmented coefficient matrix* by including the constants as an extra column on the right-hand side of the matrix

$$\left(\begin{array}{ccc|c}2 & -3 & 2 & 9\\ 3 & 2 & -1 & 4\\ 1 & -4 & 2 & 6\end{array}\right)$$

Now we operate on the rows to convert the first three columns into an upper triangular matrix

$$(1) \sim (3) \quad \begin{pmatrix}1 & -4 & 2 & 6\\ 3 & 2 & -1 & 4\\ 2 & -3 & 2 & 9\end{pmatrix} \qquad (2) \sim (3) \quad \begin{pmatrix}1 & -4 & 2 & 6\\ 2 & -3 & 2 & 9\\ 3 & 2 & -1 & 4\end{pmatrix}$$

$$\begin{array}{c}(2) - 2(1)\\ (3) - 3(1)\\ \end{array} \begin{pmatrix}1 & -4 & 2 & 6\\ 0 & 5 & -2 & -3\\ 0 & 14 & -7 & -14\end{pmatrix} \qquad \begin{array}{c}(2) \div 5\\ (3) \div 7\\ \end{array} \begin{pmatrix}1 & -4 & 2 & 6\\ 0 & 1 & -\frac{2}{5} & -\frac{3}{5}\\ 0 & 2 & -1 & -2\end{pmatrix}$$

$$(3) - 2(2) \quad \begin{pmatrix}1 & -4 & 2 & 6\\ 0 & 1 & -\frac{2}{5} & -\frac{3}{5}\\ 0 & 0 & -\frac{1}{5} & -\frac{4}{5}\end{pmatrix} \qquad (3) \times (-5) \quad \begin{pmatrix}1 & -4 & 2 & 6\\ 0 & 1 & -\frac{2}{5} & -\frac{3}{5}\\ 0 & 0 & 1 & 4\end{pmatrix}$$

The first three columns now form an upper triangular matrix which has been our purpose. If we now detach the fourth column back to its original position on the right-hand side of the matrix equation, we have

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$$\begin{pmatrix} 1 & -4 & 2 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -\frac{3}{5} \\ 4 \end{pmatrix}$$

Expanding from the bottom row, working upwards

$$\begin{aligned} x_3 &= 4 & & \therefore\ x_3 = 4 \\ x_2 - \tfrac{2}{5}x_3 &= -\tfrac{3}{5} \quad \therefore\ x_2 = -\tfrac{3}{5} + \tfrac{8}{5} = 1 & & \therefore\ x_2 = 1 \\ x_1 - 4x_2 + 2x_3 &= 6 \quad \therefore\ x_1 - 4 + 8 = 6 & & \therefore\ x_1 = 2 \end{aligned}$$

$\therefore\ x_1 = 2;\quad x_2 = 1;\quad x_3 = 4$

It is a very useful method and entails fewer tedious steps, and can be used to solve efficiently higher-order sets of equations and non-square systems. It can also solve a sequence of problems with the same coefficient matrix **A** by using the augmented matrix $(\mathbf{A}\mathbf{b}_1\mathbf{b}_2\ldots\mathbf{b}_n)$.

Example 2

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + x_4 &= -1 \\ 2x_1 - 2x_2 + x_3 - 2x_4 &= 1 \\ x_1 + x_2 - 3x_3 + x_4 &= 6 \\ 3x_1 - x_2 + 2x_3 - x_4 &= 3. \end{aligned}$$

First we write this in matrix form and compile the augmented matrix which is

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$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 2 & -2 & 1 & -2 & 1 \\ 1 & 1 & -3 & 1 & 6 \\ 3 & -1 & 2 & -1 & 3 \end{array}\right)$$

Next we operate on rows to convert the left-hand side to an upper triangular matrix. There is no set way of doing this. Use any trickery to save yourself unnecessary work.

So now you can go ahead and complete the transformations and obtain

$x_1 = \ldots\ldots\ldots\ldots;\quad x_2 = \ldots\ldots\ldots\ldots$

$x_3 = \ldots\ldots\ldots\ldots;\quad x_4 = \ldots\ldots\ldots\ldots$

$$\boxed{x_1 = 2; \quad x_2 = -3; \quad x_3 = -1; \quad x_4 = 4}$$

Here is one way. You may well have taken quite a different route.

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 2 & -2 & 1 & -2 & 1 \\ 1 & 1 & -3 & 1 & 6 \\ 3 & -1 & 2 & -1 & 3 \end{array}\right)$$

$$\begin{array}{r} \\ (2)-2(1) \\ (3)-(1) \\ (4)-[(1)+(2)] \end{array} \quad \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & -8 & 5 & -4 & 3 \\ 0 & -2 & -1 & 0 & 7 \\ 0 & -2 & 3 & 0 & 3 \end{array}\right)$$

$$\begin{array}{r} \\ (2)-4(4) \\ (3)-(4) \\ \\ \end{array} \quad \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & 0 & -7 & -4 & -9 \\ 0 & 0 & -4 & 0 & 4 \\ 0 & -2 & 3 & 0 & 3 \end{array}\right)$$

$$\begin{array}{r} \\ (2)\sim(4) \\ (3)\div 4 \\ \\ \end{array} \quad \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & -2 & 3 & 0 & 3 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -7 & -4 & -9 \end{array}\right)$$

$$\begin{array}{r} \\ (4)-7(3) \\ \\ \\ \end{array} \quad \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & -2 & 3 & 0 & 3 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -4 & -16 \end{array}\right)$$

Returning the right-hand column to its original position

$$\begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -16 \end{pmatrix}$$

Expanding from the bottom row, we have

$-4x_4 = -16$ $\qquad\therefore\ x_4 = 4$

$-x_3 = 1$ $\qquad\therefore\ x_3 = -1$

$-2x_2 + 3x_3 = 3 \quad \therefore\ -2x_2 = 6$ $\qquad\therefore\ x_2 = -3$

$x_1 + 3x_2 - 2x_3 + x_4 = -1 \quad \therefore\ x_1 - 9 + 2 + 4 = -1$ $\qquad\therefore\ x_1 = 2$

$$\therefore\ x_1 = 2; \quad x_2 = -3; \quad x_3 = -1; \quad x_4 = 4$$

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We still have a further method for solving sets of simultaneous equations.

4 Triangular decomposition method

A square matrix **A** can usually be written as a product of a lower-triangular matrix **L** and an upper-triangular matrix **U**, where $\mathbf{A} = \mathbf{LU}$.

For example, if $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 8 \\ 4 & 9 & 10 \end{pmatrix}$, **A** can be expressed as

$$\mathbf{A} = \mathbf{LU} = \underset{(\mathbf{L})}{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}} \underset{(\mathbf{U})}{\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}}$$

$$= \begin{pmatrix} l_{11}u_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + l_{22}u_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{pmatrix}$$

Note that, in **L** and **U**, elements occur in the major diagonal in each case. These are related in the product and whatever values we choose to put for u_{11}, u_{22}, $u_{33}\ldots$ then the corresponding values of l_{11}, l_{22}, $l_{33}\ldots$ will be determined – and vice versa.

For convenience, we put $u_{11} = u_{22} = u_{33}\ldots = 1$

$$\text{Then } \mathbf{A} = \mathbf{LU} = \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{pmatrix}$$

$$\text{In our example, } \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 8 \\ 4 & 9 & 10 \end{pmatrix}$$

$\therefore\ l_{11} = 1;\quad l_{11}u_{12} = 2\ \ \therefore\ u_{12} = 2;\quad l_{11}u_{13} = 3\ \ \therefore\ u_{13} = 3$

$l_{21} = 3;\quad$ Similarly $l_{22} = \ldots\ldots\ldots\ldots;\quad u_{23} = \ldots\ldots\ldots\ldots$

$l_{31} = 4;\quad l_{32} = \ldots\ldots\ldots\ldots;\quad l_{33} = \ldots\ldots\ldots\ldots$

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$$\boxed{l_{22} = -1;\quad u_{23} = 1;\quad l_{32} = 1;\quad l_{33} = -3}$$

Because

$l_{21}u_{12} + l_{22}u_{22} = 5$ that is $3 \times 2 + l_{22} \times 1 = 5$ and so $l_{22} = -1$

$l_{21}u_{13} + l_{22}u_{23} = 8$ that is $3 \times 3 + (-1) \times u_{23} = 8$ and so $u_{23} = 1$

$l_{31}u_{12} + l_{32}u_{22} = 9$ that is $4 \times 2 + l_{32} \times 1 = 9$ and so $l_{32} = 1$

$l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} = 10$ that is $4 \times 3 + 1 \times 1 + l_{33} \times 1 = 10$ and so $l_{33} = -3$

Now we substitute all these values back into the upper and lower triangular matrices and obtain

$$\mathbf{A} = \mathbf{LU} = \ldots\ldots\ldots\ldots$$

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

We have thus expressed the given matrix **A** as the product of lower and upper triangular matrices. Let us now see how we use them.

Example 1

$$\begin{aligned} x_1 + 2x_2 + \ \ 3x_3 &= 16 \\ 3x_1 + 5x_2 + \ \ 8x_3 &= 43 \\ 4x_1 + 9x_2 + 10x_3 &= 57. \end{aligned}$$

i.e. $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 5 & 8 \\ 4 & 9 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 16 \\ 43 \\ 57 \end{pmatrix}$ i.e. $\mathbf{Ax} = \mathbf{b}$.

We have seen above that **A** can be written as **LU** where

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

To solve $\mathbf{Ax} = \mathbf{b}$, we have $\mathbf{LUx} = \mathbf{b}$ i.e. $\mathbf{L}(\mathbf{Ux}) = \mathbf{b}$

Putting $\mathbf{Ux} = \mathbf{y}$, we solve $\mathbf{Ly} = \mathbf{b}$ to obtain $\mathbf{y}$

and then $\mathbf{Ux} = \mathbf{y}$ to obtain $\mathbf{x}$.

(a) Solving $\mathbf{Ly} = \mathbf{b}$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 1 & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 16 \\ 43 \\ 57 \end{pmatrix}$

Expanding from the top $y_1 = 16$; $3y_1 - y_2 = 43 \ \therefore \ y_2 = 5$; and $4y_1 + y_2 - 3y_3 = 57 \ \therefore \ 64 + 5 - 3y_3 = 57 \ \therefore \ y_3 = 4$

$$\therefore \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 16 \\ 5 \\ 4 \end{pmatrix}$$

(b) Solving $\mathbf{Ux} = \mathbf{y}$ $\quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 16 \\ 5 \\ 4 \end{pmatrix}$

Expanding from the bottom, we then have

$x_1 = \ldots\ldots\ldots\ldots$; $\quad x_2 = \ldots\ldots\ldots\ldots$; $\quad x_3 = \ldots\ldots\ldots\ldots$

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$$\boxed{x_1 = 2; \quad x_2 = 1; \quad x_3 = 4}$$

Note:

1 If $l_{ii} = 0$, then either decomposition is not possible, or, if $\mathbf{A}$ is singular, i.e. $|\mathbf{A}| = 0$, there is an infinite number of possible decompositions.

2 Instead of putting $u_{11} = u_{22} = u_{33} \ldots = 1$, we could have used the alternative substitution $l_{11} = l_{22} = l_{33} \ldots = 1$ and obtained values of u_{11}, u_{22}, $u_{33} \ldots$ etc. The working is as before.

3 One advantage of employing **LU** decomposition over Gaussian elimination is in the solution of a sequence of problems in which the same coefficient matrix occurs.

Now for another example.

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Example 2

$$x_1 + 3x_2 + 2x_3 = 19$$

$$2x_1 + x_2 + x_3 = 13$$

$$4x_1 + 2x_2 + 3x_3 = 31.$$

$$\therefore \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 19 \\ 13 \\ 31 \end{pmatrix} \quad \text{i.e. } \mathbf{Ax} = \mathbf{b}$$

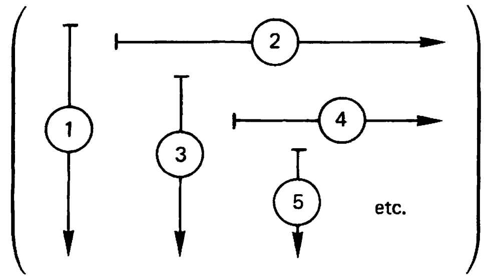
$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & 2 & 3 \end{pmatrix}$$

Now we have to find the values of the various elements. The usual order of doing this is shown by the diagram.

▶



That is, first we can write down values for l_{11}, l_{21}, l_{31} from the left-hand column; then follow this by finding u_{12}, u_{13} from the top row; and proceed for the others.

So, completing the two triangular matrices, we have

$$\mathbf{A} = \mathbf{LU} = \dots\dots\dots\dots$$

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$$\boxed{\mathbf{A} = \mathbf{LU} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 4 & -10 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & 1 \end{pmatrix}}$$

As we stated before: $\mathbf{Ax} = \mathbf{b}$; $\mathbf{L}(\mathbf{Ux}) = \mathbf{b}$. Put $\mathbf{Ux} = \mathbf{y}$

then (a) solve $\mathbf{Ly} = \mathbf{b}$ to obtain $\mathbf{y}$

and (b) solve $\mathbf{Ux} = \mathbf{y}$ to obtain $\mathbf{x}$.

Solving $\mathbf{Ly} = \mathbf{b}$ gives $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \dots \\ \dots \\ \dots \end{pmatrix}$

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$$\boxed{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 19 \\ 5 \\ 5 \end{pmatrix}}$$

Because

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 4 & -10 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 19 \\ 13 \\ 31 \end{pmatrix}$$

Expanding from the top gives

$y_1 = 19$; $y_2 = 5$; $y_3 = 5$.

(b) Now solve $\mathbf{Ux} = \mathbf{y}$ from which $x_1 = \dots\dots\dots\dots$; $x_2 = \dots\dots\dots\dots$; $x_3 = \dots\dots\dots\dots$

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$$x_1 = 3; \quad x_2 = 2; \quad x_3 = 5$$

Because we have

$$\mathbf{Ux} = y$$

i.e. $$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 19 \\ 5 \\ 5 \end{pmatrix}$$

Expanding from the bottom $x_3 = 5;\ x_2 + \dfrac{3}{5}x_3 = 5 \quad \therefore\ x_2 = 2$

and $x_1 + 3x_2 + 2x_3 = 19 \quad \therefore\ x_1 + 6 + 10 = 19 \quad \therefore\ x_1 = 3$

$$\therefore\ x_1 = 3; \quad x_2 = 2; \quad x_3 = 5$$

We can of course apply the same method to a set of four equations.

Example 3

$$\begin{aligned} x_1 + 2x_2 - \ \ x_3 + 3x_4 &= 9 \\ 2x_1 - \ \ x_2 + 3x_3 + 2x_4 &= 23 \\ 3x_1 + 3x_2 + \ \ x_3 + \ \ x_4 &= 5 \\ 4x_1 + 5x_2 - 2x_3 + 2x_4 &= -2. \end{aligned}$$

i.e. $$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 2 \\ 3 & 3 & 1 & 1 \\ 4 & 5 & -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 \\ 23 \\ 5 \\ -2 \end{pmatrix} \quad \text{i.e. } \mathbf{Ax} = \mathbf{b}$$

$$\mathbf{A} = \mathbf{LU} = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix}\begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} \\ 0 & 1 & u_{23} & u_{24} \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & -1 & 3 & 2 \\ 3 & 3 & 1 & 1 \\ 4 & 5 & -2 & 2 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} & l_{11}u_{14} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} & l_{21}u_{14} + l_{22}u_{24} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} & l_{31}u_{14} + l_{32}u_{24} + l_{33}u_{34} \\ l_{41} & l_{41}u_{12} + l_{42} & l_{41}u_{13} + l_{42}u_{23} + l_{43} & l_{41}u_{14} + l_{42}u_{24} + l_{43}u_{34} + l_{44} \end{pmatrix}$$

Now we have to find the values of the individual elements. It is easy enough if we follow the order indicated in the diagram earlier. So the two triangular matrices are

$$\mathbf{A} = \mathbf{LU} = (\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots)$$

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$$\boxed{\mathbf{A}=\mathbf{LU}=\begin{pmatrix}1&0&0&0\\2&-5&0&0\\3&-3&1&0\\4&-3&-1&-\frac{66}{5}\end{pmatrix}\begin{pmatrix}1&2&-1&3\\0&1&-1&\frac{4}{5}\\0&0&1&-\frac{28}{5}\\0&0&0&1\end{pmatrix}}$$

As usual $\mathbf{Ax}=\mathbf{b}$; $\mathbf{L}(\mathbf{Ux})=\mathbf{b}$. Put $\mathbf{Ux}=\mathbf{y}$ $\therefore$ $\mathbf{Ly}=\mathbf{b}$

(a) Solving $\mathbf{Ly}=\mathbf{b}$

$$\begin{pmatrix}1&0&0&0\\2&-5&0&0\\3&-3&1&0\\4&-3&-1&-\frac{66}{5}\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}=\begin{pmatrix}9\\23\\5\\-2\end{pmatrix}$$

$$\therefore\quad\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}=\begin{pmatrix}\dots\\\dots\\\dots\\\dots\end{pmatrix}$$

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$$\boxed{\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}=\begin{pmatrix}9\\-1\\-25\\5\end{pmatrix}}$$

(b) Solving $\mathbf{Ux}=\mathbf{y}$

$$\begin{pmatrix}1&2&-1&3\\0&1&-1&\frac{4}{5}\\0&0&1&-\frac{28}{5}\\0&0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}9\\-1\\-25\\5\end{pmatrix}$$

which finally gives

$x_1=\dots\dots\dots\dots$; $x_2=\dots\dots\dots\dots$

$x_3=\dots\dots\dots\dots$; $x_4=\dots\dots\dots\dots$

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$$x_1 = 1; \quad x_2 = -2; \quad x_3 = 3; \quad x_4 = 5$$

Comparison of methods

Inverse method

This is an elementary method but it is very inefficient when the number of equations to solve increases beyond three.

Row transformation method

An efficient method but each case is different and relies on ingenuity to see the way forward.

Gaussian elimination method

The most efficient method and should be used in most cases. It must be used when there is a singular or non-square system.

Triangular decomposition method

An alternative to Gaussian elimination in some cases.

Now let us proceed to something rather different, so move on to the next frame for a new start

Eigenvalues and eigenvectors

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Matrices commonly appear in technological problems, for example those involving coupled oscillations and vibrations, and give rise to equations of the form

$$\mathbf{Ax} = \lambda\mathbf{x}$$

where $\mathbf{A} = (a_{ij})$ is a square matrix, $\mathbf{x}$ is a column matrix (x_i) and λ is a scalar quantity, i.e. a number.

For non-trivial solutions, i.e. for $\mathbf{x} \neq \mathbf{0}$, the values of λ are called the *eigenvalues*, *characteristic values*, or *latent roots* of the matrix $\mathbf{A}$ and the corresponding solutions of the given equations $\mathbf{Ax} = \lambda\mathbf{x}$ are called the *eigenvectors*, or *characteristic vectors* of $\mathbf{A}$ (refer to *Engineering Mathematics (Fifth Edition)*, pages 558ff).

▶

The set of equations

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

then simplifies to

$$\begin{pmatrix} (a_{11}-\lambda) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{22}-\lambda) & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & (a_{nn}-\lambda) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

That is, $\mathbf{Ax} = \lambda\mathbf{x}$ becomes $\mathbf{Ax} - \lambda\mathbf{x} = \mathbf{0}$

i.e. $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$

the unit matrix $\mathbf{I}$ being introduced since we can subtract only a matrix from another matrix.

For this set of homogeneous linear equations (right-hand side constant terms all zero) to have non-trivial solutions

$|\mathbf{A} - \lambda\mathbf{I}|$ **must be zero**

This is called the *characteristic determinant* of $\mathbf{A}$ and $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is the *characteristic equation*, the solution of which gives the values of λ, i.e. the eigenvalues of $\mathbf{A}$.

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Example 1

Find the eigenvalues and corresponding eigenvectors of

$\mathbf{Ax} = \lambda\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$.

The characteristic equation is $|\mathbf{A} - \lambda\mathbf{I}| = 0$

i.e. $\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = 0$, which, when expanded, gives

$\lambda_1 = \dots\dots\dots\dots$ and $\lambda_2 = \dots\dots\dots\dots$

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$$\boxed{\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 5}$$

Because

$(2-\lambda)(1-\lambda) - 12 = 0 \quad \therefore \; 2 - 3\lambda + \lambda^2 - 12 = 0$

$\lambda^2 - 3\lambda - 10 = 0 \quad (\lambda-5)(\lambda+2) = 0 \qquad \therefore \; \lambda = -2 \text{ or } 5$

Now we substitute each value of λ in turn in the equation

$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$

With $\lambda = -2$

$$\left\{\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - (-2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Multiplying out the left-hand side, we get

....

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$$\boxed{4x_1 + 3x_2 = 0}$$

from which we get $x_2 = -\frac{4}{3}x_1$ i.e. not specific values for x_1 and x_2, but a relationship between them. Whatever value we assign to x_1 we obtain a corresponding value of x_2.

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \text{ or } \begin{pmatrix} 6 \\ -8 \end{pmatrix} \text{ or } \begin{pmatrix} 9 \\ -12 \end{pmatrix}, \quad \text{etc.}$$

The most convenient way to do this is to choose $x_1 = 1$ and then scale $\mathbf{x}_1$ to obtain integer elements. So here we find for $x_1 = 1$ then $x_2 = -4/3$ so $\mathbf{x}_1$ is of the form

$$\begin{pmatrix} 1 \\ -\dfrac{4}{3} \end{pmatrix}$$

This is now scaled up by multiplying by 3 to give

$\mathbf{x}_1 = \alpha\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ where α is a constant multiplier.

The simplest result, with $\alpha = 1$, is the one normally quoted.

$$\therefore \text{ for } \lambda_1 = -2, \quad \mathbf{x}_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Similarly, for $\lambda_2 = 5$, the corresponding eigenvector is ....

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$$\boxed{\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Because, with $\lambda_2 = 5$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ becomes

$$\left\{\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore \quad -3x_1 + 3x_2 = 0 \quad$ i.e. $x_2 = x_1$

$\therefore \quad$ with $\lambda_2 = 5$, the corresponding eigenvector is $\mathbf{x}_2 = \beta\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Again, taking $\beta = 1$, for $\lambda_2 = 5$, $\quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So the required eigenvectors are

$\mathbf{x}_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \quad$ corresponding to $\lambda_1 = -2$

$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad$ corresponding to $\lambda_2 = 5$.

Example 2

Determine the eigenvalues and corresponding eigenvectors of

$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}$.

The characteristic equation is $|\mathbf{A} - \lambda\mathbf{I}| = 0$, which in this case can be written as ....

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$$\boxed{\begin{vmatrix} 3-\lambda & 10 \\ 2 & 4-\lambda \end{vmatrix} = 0}$$

Expanding the determinant and solving the equation gives

$\lambda_1 = \ldots\ldots\ldots\ldots; \quad \lambda_2 = \ldots\ldots\ldots\ldots$

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$$\boxed{\lambda_1 = -1; \quad \lambda_2 = 8}$$

Because the equation is $(3-\lambda)(4-\lambda) - 20 = 0 \quad \therefore \ \lambda^2 - 7\lambda - 8 = 0$

$\therefore \ (\lambda+1)(\lambda-8) = 0 \quad \therefore \ \lambda = -1$ or 8

(a) With $\lambda_1 = -1$, we solve $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ to obtain an eigenvector, which is ....

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$$\boxed{\mathbf{x}_1 = \begin{pmatrix} 5 \\ -2 \end{pmatrix}}$$

Because

$$\mathbf{A} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} \quad \therefore \ \left\{ \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 10 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore \ 4x_1 + 10x_2 = 0 \quad \therefore \ x_2 = -\frac{2}{5}x_1 \quad \mathbf{x_1} = \alpha\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$\therefore$ with $\alpha = 1 \quad \lambda_1 = -1$ and $\mathbf{x_1} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

(b) In the same way the corresponding eigenvector $\mathbf{x_2}$ for $\lambda_2 = 8$ is

....

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$$\boxed{\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

Because

$$\left\{ \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} - 8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 10 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore \ -5x_1 + 10x_2 = 0 \quad \therefore \ x_2 = \frac{1}{2}x_1 \quad \mathbf{x}_2 = \beta\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore$ with $\beta = 1, \quad \lambda_2 = 8$ and $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

▶

The same basic method can similarly be applied to third-order sets of equations.

Example 3

Determine the eigenvalues and eigenvectors of $\mathbf{Ax} = \lambda\mathbf{x}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix}.$$

As before, we have $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with characteristic equation $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

$$\text{i.e.} \begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 3 & 1 & -3-\lambda \end{vmatrix} = 0$$

Expanding this we have

$\lambda_1 = \ldots\ldots\ldots\ldots;\quad \lambda_2 = \ldots\ldots\ldots\ldots;\quad \lambda_3 = \ldots\ldots\ldots\ldots$

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$$\boxed{\lambda_1 = 2;\quad \lambda_2 = 3;\quad \lambda_3 = -5}$$

Because

$$(1-\lambda)\{(2-\lambda)(-3-\lambda) - 0\} + 4\{0 - 3(2-\lambda)\} = 0$$
$$(1-\lambda)(2-\lambda)(-3-\lambda) - 12(2-\lambda) = 0$$
$$\therefore\ (2-\lambda)\{(1-\lambda)(-3-\lambda) - 12\} = 0$$
$$\therefore\ \lambda = 2 \ \text{ or } \ \lambda^2 + 2\lambda - 15 = 0 \quad \therefore\ (\lambda-3)(\lambda+5) = 0$$
$$\therefore\ \lambda = 2,\ 3, \text{ or } -5$$

(a) With $\lambda_1 = 2$, $\quad(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ becomes

$$\left\{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore\ \begin{pmatrix} -1 & 0 & 4 \\ 0 & 0 & 0 \\ 3 & 1 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which a corresponding eigenvector $\mathbf{x}_1$ is $\ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{x}_1 = \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix}}$$

Because we have $-x_1 + 4x_3 = 0 \qquad \therefore\ x_3 = \frac{1}{4}x_1$

$3x_1 + x_2 - 5x_3 = 0 \quad \therefore\ 3x_1 + x_2 - \frac{5}{4}x_1 = 0 \quad \therefore\ x_2 = -\frac{7}{4}x_1$

$\therefore\ x_1, x_2, x_3$ are in the ratio $1 : -\dfrac{7}{4} : \dfrac{1}{4}$ i.e. $4 : -7 : 1 \quad \therefore\ \mathbf{x}_1 = \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix}$

(b) Similarly for $\lambda_2 = 3$, $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$

$$\left\{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

from which a corresponding eigenvector is

$\mathbf{x}_2 = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}$$

Because

$$\left\{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & 4 \\ 0 & -1 & 0 \\ 3 & 1 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore\ -2x_1 + 4x_3 = 0 \quad \therefore\ x_3 = \frac{1}{2}x_1$

Also $\qquad -x_2 = 0 \quad \therefore\ x_2 = 0 \qquad \therefore\ \mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

(c) All that now remains is $\lambda_3 = -5$. A corresponding eigenvector $\mathbf{x}_3$ is

$\mathbf{x}_3 = \ldots\ldots\ldots\ldots$

Finish it on your own. Method just the same as before.

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$$\boxed{\mathbf{x_3} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}}$$

Check the working.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} \quad \text{and } \lambda_3 = -5 \text{ with } (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

$$\left\{\begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 0 & 4 \\ 0 & 7 & 0 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore\ 6x_1 + 4x_3 = 0 \quad \therefore\ x_3 = -\tfrac{3}{2}x_1$$

$$7x_2 = 0 \quad \therefore\ x_2 = 0 \qquad \therefore\ \mathbf{x_3} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$$

Collecting the results together, we finally have

$$\lambda_1 = 2,\ \mathbf{x_1} = \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix};\ \lambda_2 = 3,\ \mathbf{x_2} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix};\ \lambda_3 = -5,\ \mathbf{x_3} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$$

Cayley–Hamilton theorem

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The Cayley–Hamilton theorem states that every square matrix satisfies its characteristic equation. For example the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

of Frame 54 has the characteristic equation

$$\lambda^2 - 3\lambda - 10 = 0$$

and so the Cayley–Hamilton theorem tells us that

$$\mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I} = \mathbf{0}$$

▶

To verify this we note that

$$\mathbf{A}^2 = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix} \text{ so that}$$

$$\mathbf{A}^2 - 3\mathbf{A} - 10\mathbf{I} = \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix} - 3\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} - 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix} - \begin{pmatrix} 6 & 9 \\ 12 & 3 \end{pmatrix} - \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

You try one. Verify that the matrix $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}$ of Frame 57 with the characteristic equation

$$\lambda^2 - 7\lambda - 8 = 0$$

satisfies the Cayley–Hamilton theorem, that is ....

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$$\boxed{\mathbf{A}^2 - 7\mathbf{A} - 8\mathbf{I} = \mathbf{0}}$$

Because

$$\mathbf{A}^2 = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 29 & 70 \\ 14 & 36 \end{pmatrix} \text{ so that}$$

$$\mathbf{A}^2 - 7\mathbf{A} - 8\mathbf{I} = \begin{pmatrix} 29 & 70 \\ 14 & 36 \end{pmatrix} - 7\begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} - 8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 29 & 70 \\ 14 & 36 \end{pmatrix} - \begin{pmatrix} 21 & 70 \\ 14 & 28 \end{pmatrix} - \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Now on to something different

Systems of first-order ordinary differential equations

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Matrix methods involving eigenvalues and their associated eigenvectors can be used to solve systems of coupled differential equations, though we shall only consider cases where the relevant eigenvalues are distinct. We proceed by example.

Example 1

Consider the system of two coupled ordinary differential equations

$$\begin{aligned} f_1'(x) &= 2f_1(x) + 3f_2(x) \\ f_2'(x) &= 4f_1(x) + f_2(x) \end{aligned} \quad \text{where } f_1(0) = 2 \text{ and } f_2(0) = 1$$

These can be written in matrix form as ....

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$$\boxed{\begin{pmatrix} f_1'(x) \\ f_2'(x) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}}$$

That is

$\mathbf{F}'(x) = \mathbf{AF}(x)$

where $\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $\mathbf{F}'(x) = \begin{pmatrix} f_1'(x) \\ f_2'(x) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ and where $\mathbf{F}(0) = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are the boundary conditions in matrix form.

The matrix differential equation $\mathbf{F}'(x) = \mathbf{AF}(x)$ is similar in form to the single differential equation $f'(x) = af(x)$ (a constant) which has solution $f(x) = \alpha e^{ax}$ (α constant), so to solve the matrix equation we try a solution of the form

$\mathbf{F}(x) = \mathbf{C}e^{kx}$ where the number k and the constants c_1 and c_2 of the matrix $\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ are to be determined.

Substituting $\mathbf{F}(x) = \mathbf{C}e^{kx}$ into the matrix equation $\mathbf{F}'(x) = \mathbf{AF}(x)$ gives

....

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$$\boxed{k\mathbf{C}e^{kx} = \mathbf{AC}e^{kx}}$$

Because

$\mathbf{F}(x) = \mathbf{C}e^{kx}$ so $\mathbf{F}'(x) = k\mathbf{C}e^{kx}$. Since $\mathbf{F}'(x) = \mathbf{AF}(x)$ then $k\mathbf{C}e^{kx} = \mathbf{AC}e^{kx}$

Dividing both sides by e^{kx} gives

$k\mathbf{C} = \mathbf{AC}$ that is $\mathbf{AC} = k\mathbf{C}$.

So, from Frame 53, k is an *eigenvalue* of $\mathbf{A}$ and $\mathbf{C}$ is the corresponding *eigenvector*. Therefore, we must first find the eigenvalues of $\mathbf{A}$ and for this matrix these have been found earlier in Frames 54 to 57. They are

$\lambda = -2$ (and so $k = -2$) with corresponding eigenvector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$\lambda = 5$ (and so $k = 5$) with corresponding eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

To each eigenvalue the matrix $\mathbf{F}(x) = \mathbf{C}e^{kx}$ is a solution. The complete solution to $\mathbf{F}' = \mathbf{AF}$ is then

$$\mathbf{F}_1(x) = \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} a_1 e^{\cdots} \quad \text{and} \quad \mathbf{F}_2(x) = \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} a_2 e^{\cdots}$$

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$$\mathbf{F}_1(x) = \begin{pmatrix} 3 \\ -4 \end{pmatrix} a_1 e^{-2x} \text{ and } \mathbf{F}_2(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} a_2 e^{5x}$$

Because

$\mathbf{F}(x) = \mathbf{C}e^{kx}$ is the solution corresponding to the eigenvalue k with associated eigenvector $\mathbf{C}$.

The complete solution to the equation $\mathbf{F}'(x) = \mathbf{AF}(x)$ is then a combination of these two solutions in the form

$$\mathbf{F}(x) = A\begin{pmatrix} 3 \\ -4 \end{pmatrix} e^{-2x} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5x}$$

Applying the boundary conditions gives $\mathbf{F}(0) = \dots\dots\dots\dots$

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$$\mathbf{F}(0) = \begin{pmatrix} 3A + B \\ -4A + B \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Because

$$\mathbf{F}(x) = A\begin{pmatrix} 3 \\ -4 \end{pmatrix} e^{-2x} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5x} \text{ and so } \mathbf{F}(0) = A\begin{pmatrix} 3 \\ -4 \end{pmatrix} + B\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3A + B \\ -4A + B \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore

$$\begin{aligned} 3A + B &= 2 \\ -4A + B &= 1 \end{aligned} \quad \text{with solution } A = 1/7 \text{ and } B = 11/7,$$

giving the final solution as $\mathbf{F}(x) = \dots\dots\dots\dots$

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$$\mathbf{F}(x) = \begin{pmatrix} 3/7 \\ -4/7 \end{pmatrix} e^{-2x} + \begin{pmatrix} 11/7 \\ 11/7 \end{pmatrix} e^{5x}$$

Summary

To solve an equation of the form

$\mathbf{F}'(x) = \mathbf{AF}(x)$

1 Find the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $\mathbf{A}$ (assuming they are all distinct)

2 Find the associated eigenvectors $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_n$

3 Write the solution of the equation as $\mathbf{F}(x) = \sum_{r=1}^{n} (A_r e^{\lambda_r x})\mathbf{C}_r$ and use the boundary conditions to find the values of a_r for $r = 1, 2, \dots, n$.

Now you try one.

Next frame

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Example 2

The system of two coupled ordinary differential equations

$$\begin{aligned} f_1'(x) &= 3f_1(x) + 10f_2(x) \\ f_2'(x) &= 2f_1(x) + 4f_2(x) \end{aligned} \quad \text{where } f_1(0) = 0 \text{ and } f_2(0) = 1$$

has the solution (refer to Frames 57 to 61)

$$\begin{aligned} f_1(x) &= \ldots\ldots\ldots\ldots \\ f_2(x) &= \ldots\ldots\ldots\ldots \end{aligned}$$

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$$\boxed{\begin{aligned} f_1(x) &= -\frac{10}{9}e^{-x} + \frac{10}{9}e^{8x} \\ f_2(x) &= \frac{4}{9}e^{-x} + \frac{5}{9}e^{8x} \end{aligned}}$$

Because

$$\begin{aligned} f_1'(x) &= 3f_1(x) + 10f_2(x) \\ f_2'(x) &= 2f_1(x) + 4f_2(x) \end{aligned}$$

can be written in matrix form as

$$\ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{pmatrix} f_1'(x) \\ f_2'(x) \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}}$$

That is

$$\mathbf{F}'(x) = \mathbf{AF}(x)$$

where $\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $\mathbf{F}'(x) = \begin{pmatrix} f_1'(x) \\ f_2'(x) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}$ and where $\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

To solve the matrix equation we first need the eigenvalues and associated eigenvectors of the matrix **A**. These have already been found in Frames 57 to 61 and they are

$\lambda = -1$ with corresponding eigenvector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$\lambda = 8$ with corresponding eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The complete solution of $\mathbf{F}' = \mathbf{AF}$ is then

$$\mathbf{F}(x) = A\begin{pmatrix} 5 \\ -2 \end{pmatrix}e^{-x} + B\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{8x}$$

That is $f_1(x) = \ldots\ldots\ldots\ldots$

$f_2(x) = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{aligned} f_1(x) &= 5Ae^{-x} + 2Be^{8x} \\ f_2(x) &= -2Ae^{-x} + Be^{8x} \end{aligned}}$$

Because

$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = A\begin{pmatrix} 5 \\ -2 \end{pmatrix}e^{-x} + B\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{8x}$$

and so

$$f_1(x) = 5Ae^{-x} + 2Be^{8x}$$
$$f_2(x) = -2Ae^{-x} + Be^{8x}$$

Applying the boundary conditions, we find

$$\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \ldots A + \ldots B \\ \ldots A + \ldots B \end{pmatrix}$$

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$$\boxed{\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 5A + 2B \\ -2A + B \end{pmatrix}}$$

Because

The boundary conditions are $f_1(0) = 0$ and $f_2(0) = 1$ therefore

$$\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 5A + 2B \\ -2A + B \end{pmatrix}$$

This gives the pair of simultaneous equations

$$\begin{aligned} 5A + 2B &= 0 \\ -2A + B &= 1 \end{aligned} \quad \text{which have solution}$$

$A = \ldots\ldots\ldots\ldots$ and $B = \ldots\ldots\ldots\ldots$

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$$\boxed{A = -2/9 \text{ and } B = 5/9}$$

This gives the complete solution as

$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} -10/9 \\ 4/9 \end{pmatrix}e^{-x} + \begin{pmatrix} 10/9 \\ 5/9 \end{pmatrix}e^{8x}$$

$$f_1(x) = -\frac{10}{9}e^{-x} + \frac{10}{9}e^{8x}$$

$$f_2(x) = \frac{4}{9}e^{-x} + \frac{5}{9}e^{8x}$$

Diagonalisation of a matrix

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Modal matrix

We have already discussed the eigenvalues and eigenvectors of a matrix $\mathbf{A}$ of order n. In this section we shall assume that all the eigenvalues are distinct. If the n eigenvectors $\mathbf{x}_i$ are arranged as columns of a square matrix, the *modal matrix* of $\mathbf{A}$, denoted by $\mathbf{M}$, is formed

i.e. $\mathbf{M} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n)$

For example, we have seen earlier that if

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} \text{ then } \lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = -5$$

and the corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 4 \\ -7 \\ 1 \end{pmatrix},\ \mathbf{x}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix},\ \mathbf{x}_3 = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$$

Then the modal matrix $\mathbf{M} = \begin{pmatrix} 4 & 2 & 2 \\ -7 & 0 & 0 \\ 1 & 1 & -3 \end{pmatrix}$

Spectral matrix

Also, we define the *spectral matrix* of $\mathbf{A}$, i.e. $\mathbf{S}$, as a diagonal matrix with the eigenvalues only on the main diagonal

$$\text{i.e. } \mathbf{S} = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

So, in the example above, $\mathbf{S} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{S} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}}$$

Note that the eigenvalues of $\mathbf{S}$ and $\mathbf{A}$ are the same.

So, if $\mathbf{A} = \begin{pmatrix} 5 & -6 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ has eigenvalues $\lambda = 1, 2, 4$ and

corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$

then $\mathbf{M} = \ldots\ldots\ldots\ldots$ and $\mathbf{S} = \ldots\ldots\ldots\ldots$

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$$\mathbf{M}=\begin{pmatrix}0&1&3\\1&1&1\\6&3&3\end{pmatrix};\quad \mathbf{S}=\begin{pmatrix}1&0&0\\0&2&0\\0&0&4\end{pmatrix}$$

Now how are these connected? Let us investigate.

The eigenvectors $\mathbf{x}$ arranged in the modal matrix satisfy the original equation

$$\mathbf{Ax}=\lambda\mathbf{x}$$

Also $\quad\mathbf{M}=(\mathbf{x}_1\ \ \mathbf{x}_2\ \ \ldots\ \ \mathbf{x}_n)$

Then $\quad\mathbf{AM}=\mathbf{A}(\mathbf{x}_1\ \ \mathbf{x}_2\ \ \ldots\ \ \mathbf{x}_n)$

$$=(\mathbf{Ax}_1\ \ \mathbf{Ax}_2\ \ \ldots\ \ \mathbf{Ax}_n)$$

$$=(\lambda_1\mathbf{x}_1\ \ \lambda_2\mathbf{x}_2\ \ \ldots\ \ \lambda_n\mathbf{x}_n)\quad \text{since } \mathbf{Ax}=\lambda\mathbf{x}$$

Now $\quad\mathbf{S}=\begin{pmatrix}\lambda_1&0&\ldots&0\\0&\lambda_2&\ldots&0\\\vdots&\vdots&&\vdots\\0&0&\ldots&\lambda_n\end{pmatrix}\quad\therefore\ (\lambda_1\mathbf{x}_1\ \ \lambda_2\mathbf{x}_2\ \ \ldots\ \ \lambda_n\mathbf{x}_n)=\mathbf{MS}$

$\therefore\ \mathbf{AM}=\mathbf{MS}$

If we now pre-multiply both sides by $\mathbf{M}^{-1}$ we have

$$\mathbf{M}^{-1}\mathbf{AM}=\mathbf{M}^{-1}\mathbf{MS}\qquad\text{But }\ \mathbf{M}^{-1}\mathbf{M}=\mathbf{I}$$

$$\therefore\ \mathbf{M}^{-1}\mathbf{AM}=\mathbf{S}$$

Make a note of this result. Then we will consider an example

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Example 1

From the results of a previous example in Frame 65, if

$$\mathbf{A}=\begin{pmatrix}1&0&4\\0&2&0\\3&1&-3\end{pmatrix}\ \text{ then }\ \lambda_1=2,\ \lambda_2=3,\ \lambda_3=-5\ \text{ and}$$

$$\mathbf{x}_1=\begin{pmatrix}4\\-7\\1\end{pmatrix},\quad \mathbf{x}_2=\begin{pmatrix}2\\0\\1\end{pmatrix},\quad \mathbf{x}_3=\begin{pmatrix}2\\0\\-3\end{pmatrix}.$$

Also $\mathbf{M}=\begin{pmatrix}4&2&2\\-7&0&0\\1&1&-3\end{pmatrix}$.

We can find $\mathbf{M}^{-1}$ by any of the methods we have established previously.

$$\mathbf{M}^{-1}=\ldots\ldots\ldots\ldots$$

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$$\boxed{\mathbf{M}^{-1} = \begin{pmatrix} 0 & -1/7 & 0 \\ 3/8 & 1/4 & 1/4 \\ 1/8 & 1/28 & -1/4 \end{pmatrix}}$$

Here is one way of determining the inverse. You may have done it by another.

$$\left(\begin{array}{ccc|ccc} 4 & 2 & 2 & 1 & 0 & 0 \\ -7 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & -3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 7 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & -3 & 0 & 0 & 1 \\ 4 & 2 & 2 & 1 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/7 & 0 \\ 0 & 1 & -3 & 0 & 1/7 & 1 \\ 0 & 2 & 2 & 1 & 4/7 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/7 & 0 \\ 0 & 1 & -3 & 0 & 1/7 & 1 \\ 0 & 0 & 8 & 1 & 2/7 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/7 & 0 \\ 0 & 1 & -3 & 0 & 1/7 & 1 \\ 0 & 0 & 1 & 1/8 & 1/28 & -1/4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1/7 & 0 \\ 0 & 1 & 0 & 3/8 & 7/28 & 1/4 \\ 0 & 0 & 1 & 1/8 & 1/28 & -1/4 \end{array}\right)$$

$$\therefore \mathbf{M}^{-1} = \begin{pmatrix} 0 & -1/7 & 0 \\ 3/8 & 1/4 & 1/4 \\ 1/8 & 1/28 & -1/4 \end{pmatrix}$$

So now $\mathbf{A} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix}$ and $\mathbf{M} = \begin{pmatrix} 4 & 2 & 2 \\ -7 & 0 & 0 \\ 1 & 1 & -3 \end{pmatrix}$

$$\therefore \mathbf{AM} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix} \begin{pmatrix} 4 & 2 & 2 \\ -7 & 0 & 0 \\ 1 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 6 & -10 \\ -14 & 0 & 0 \\ 2 & 3 & 15 \end{pmatrix}$$

Then $\mathbf{M}^{-1}\mathbf{AM} = \begin{pmatrix} 0 & -1/7 & 0 \\ 3/8 & 1/4 & 1/4 \\ 1/8 & 1/28 & -1/4 \end{pmatrix} \begin{pmatrix} 8 & 6 & -10 \\ -14 & 0 & 0 \\ 2 & 3 & 15 \end{pmatrix}$

$= \ldots\ldots\ldots\ldots$

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$$\mathbf{M}^{-1}\mathbf{AM} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

So we have transformed the original matrix **A** into a diagonal matrix and notice that the elements on the main diagonal are, in fact, the eigenvalues of **A**

i.e. $\mathbf{M}^{-1}\mathbf{AM} = \mathbf{S}$

Therefore, let us list a few relevant facts

1. $\mathbf{M}^{-1}\mathbf{AM}$ transforms the square matrix **A** into a diagonal matrix **S**.
2. A square matrix **A** of order n can be so transformed if the matrix has n independent eigenvectors.
3. A matrix **A** always has n linearly independent eigenvectors if it has n distinct eigenvalues or if it is a symmetric matrix.
4. If the matrix has repeated eigenvalues and is not symmetric, it may or may not have n linearly independent eigenvectors.

Now here is one straightforward example with which to finish.

Example 2

If $\mathbf{A} = \begin{pmatrix} -6 & 5 \\ 4 & 2 \end{pmatrix}$, $\quad \mathbf{M} = \dots\dots\dots\dots$; $\mathbf{M}^{-1} = \dots\dots\dots\dots$;

and hence $\mathbf{M}^{-1}\mathbf{AM} = \dots\dots\dots\dots$

Work through it entirely on your own:

(1) Determine the eigenvalues and corresponding eigenvectors.

(2) Hence form the matrix **M**.

(3) Determine $\mathbf{M}^{-1}$, the inverse of **M**.

(4) Finally form the matrix products **AM** and $\mathbf{M}^{-1}(\mathbf{AM})$.

$$\mathbf{M} = \begin{pmatrix} 1 & 5 \\ 2 & -2 \end{pmatrix}; \quad \mathbf{M}^{-1} = \begin{pmatrix} 1/6 & 5/12 \\ 1/6 & -1/12 \end{pmatrix}; \quad \mathbf{M}^{-1}\mathbf{AM} = \begin{pmatrix} 4 & 0 \\ 0 & -8 \end{pmatrix}$$

Here is the working. See whether you agree.

$$\mathbf{A} = \begin{pmatrix} -6 & 5 \\ 4 & 2 \end{pmatrix} \quad \therefore \begin{vmatrix} -6-\lambda & 5 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$$(-6-\lambda)(2-\lambda) - 20 = 0 \quad \therefore \lambda^2 + 4\lambda - 32 = 0$$

$$(\lambda - 4)(\lambda + 8) = 0 \quad \therefore \lambda = 4 \text{ or } -8$$

(a) $\lambda_1 = 4$ $\quad \left\{ \begin{pmatrix} -6 & 5 \\ 4 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -10 & 5 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore -10x_1 + 5x_2 = 0 \quad \therefore x_2 = 2x_1 \qquad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(b) $\lambda_2 = -8$ $\quad \left\{ \begin{pmatrix} -6 & 5 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \right\} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 & 5 \\ 4 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore 2x_1 + 5x_2 = 0 \quad \therefore x_2 = -\tfrac{2}{5}x_1 \quad \therefore \mathbf{x}_2 = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$$\therefore \mathbf{M} = \begin{pmatrix} 1 & 5 \\ 2 & -2 \end{pmatrix}$$

To find $\mathbf{M}^{-1}$ $\quad \left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{array}\right)$

Operating on rows, we have

$$\left(\begin{array}{cc|cc} 0 & 5 & 1 & 0 \\ 0 & -12 & -2 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 1 & 1/6 & -1/12 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & 1/6 & 5/12 \\ 0 & 1 & 1/6 & -1/12 \end{array}\right)$$

$$\therefore \mathbf{M}^{-1} = \begin{pmatrix} 1/6 & 5/12 \\ 1/6 & -1/12 \end{pmatrix}$$

$$\therefore \mathbf{AM} = \begin{pmatrix} -6 & 5 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 4 & -40 \\ 8 & 16 \end{pmatrix}$$

$$\therefore \mathbf{M}^{-1}\mathbf{AM} = \begin{pmatrix} 1/6 & 5/12 \\ 1/6 & -1/12 \end{pmatrix} \begin{pmatrix} 4 & -40 \\ 8 & 16 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -8 \end{pmatrix}$$

$$\therefore \mathbf{M}^{-1}\mathbf{AM} = \begin{pmatrix} 4 & 0 \\ 0 & -8 \end{pmatrix}$$

Systems of second-order differential equations

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The process of uncoupling a system of differential equations to obtain their solution can be achieved by diagonalising the matrix of coefficients. For simplicity we shall only consider second-order equations and again, we proceed by example.

Example 1

Consider the system of coupled second-order differential equations

$$f_1''(x) = 2f_1(x) + 3f_2(x)$$
$$f_2''(x) = 4f_1(x) + f_2(x)$$

where $f_1(0) = 2$, $f_2(0) = 1$, $f_1'(0) = 4$ and $f_2'(0) = 3$

These can be written in matrix form as

. . . .

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$$\boxed{\begin{pmatrix} f_1''(x) \\ f_2''(x) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}}$$

That is

$$\mathbf{F}''(x) = \mathbf{AF}(x)$$

where $\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $\mathbf{F}''(x) = \begin{pmatrix} f_1''(x) \\ f_2''(x) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ and where $\mathbf{F}(0) = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{F}'(0) = \begin{pmatrix} f_1'(0) \\ f_2'(0) \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ are the boundary conditions in matrix form.

The matrix differential equation $\mathbf{F}''(x) = \mathbf{AF}(x)$ is similar in form to the single differential equation $f''(x) = af(x)$ (a constant) which has solution $f(x) = \alpha e^{\sqrt{a}x} + \beta e^{-\sqrt{a}x}$ (α, β constants), so to solve the matrix equation we try a solution of this form. We already know from Frames 54 to 57 that the eigenvalues and eigenvectors of matrix $\mathbf{A}$ are

$\lambda = -2$ with corresponding eigenvector $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$\lambda = 5$ with corresponding eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The modal matrix of $\mathbf{A}$ is the matrix $\mathbf{M}$ and the spectral matrix of $\mathbf{A}$ is the matrix $\mathbf{S}$ where

$$\mathbf{M} = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$$

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$$\mathbf{M} = \begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$$

Because

The modal matrix is formed from the eigenvectors of $\mathbf{A}$. That is

$\mathbf{M} = \begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix}$ where the two eigenvectors are $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The spectral matrix is formed from the eigenvalues of $\mathbf{A}$. That is

$\mathbf{S} = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$ where the two eigenvalues are -2 and 5

If we now define the matrix $\mathbf{G}(x)$ by the equation $\mathbf{F}(x) = \mathbf{MG}(x)$, then differentiating gives

$\mathbf{F}''(x) = [\mathbf{MG}(x)]'' = \mathbf{MG}''(x)$ where

$\mathbf{F}''(x) = \mathbf{AF}(x) = \mathbf{AMG}(x)$

and so, from Frame 85, $\mathbf{M}^{-1}\mathbf{MG}''(x) = \mathbf{G}''(x) = \mathbf{M}^{-1}\mathbf{AMG}(x) = \mathbf{SG}(x)$. That is

$\mathbf{G}''(x) = \mathbf{SG}(x)$

Therefore, in component terms

$$\mathbf{G}''(x) = \begin{pmatrix} g_1''(x) \\ g_2''(x) \end{pmatrix} = \mathbf{SG}(x) = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$$

and so

$g_1''(x) = \ldots g_1(x)$ with solution $g_1(x) = k_{11}e^{\cdots x} + k_{12}e^{-\cdots x}$

$g_2''(x) = \ldots g_2(x)$ with solution $g_2(x) = k_{21}e^{\cdots x} + k_{22}e^{-\cdots x}$

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$g_1''(x) = -2g_1(x)$ with solution $g_1(x) = k_{11}e^{j\sqrt{2}x} + k_{12}e^{-j\sqrt{2}x}$

$g_2''(x) = 5g_2(x)$ with solution $g_2(x) = k_{21}e^{\sqrt{5}x} + k_{22}e^{-\sqrt{5}x}$

Now, $\mathbf{F}(x) = \mathbf{MG}(x)$ so

$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} \ldots \\ \ldots \end{pmatrix}$$

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$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \begin{pmatrix} 3k_{11}e^{j\sqrt{2}x} + 3k_{12}e^{-j\sqrt{2}x} + k_{21}e^{\sqrt{5}x} + k_{22}e^{-\sqrt{5}x} \\ -4k_{11}e^{j\sqrt{2}x} - 4k_{12}e^{-j\sqrt{2}x} + k_{21}e^{\sqrt{5}x} + k_{22}e^{-\sqrt{5}x} \end{pmatrix}$$

Because

$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} = \mathbf{MG}(x) = \begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} k_{11}e^{j\sqrt{2}x} + k_{12}e^{-j\sqrt{2}x} \\ k_{21}e^{\sqrt{5}x} + k_{22}e^{-\sqrt{5}x} \end{pmatrix}$$

and so

$$f_1(x) = 3k_{11}e^{j\sqrt{2}x} + 3k_{12}e^{-j\sqrt{2}x} + k_{21}e^{j\sqrt{5}x} + k_{22}e^{-\sqrt{5}x}$$

and

$$f_2(x) = -4k_{11}e^{j\sqrt{2}x} - 4k_{12}e^{-j\sqrt{2}x} + k_{21}e^{\sqrt{5}x} + k_{22}e^{-\sqrt{5}x}$$

This solution can be written in terms of circular and hyperbolic trigonometric expressions as

$$\mathbf{F}(x) = \begin{pmatrix} \dots & \dots \\ \dots & \dots \end{pmatrix} \begin{pmatrix} P\cos \dots x + Q\sin \dots x \\ R\cosh \dots x + S\sinh \dots x \end{pmatrix}$$

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$$\mathbf{F}(x) = \begin{pmatrix} 3 & 1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} P\cos\sqrt{2}x + Q\sin\sqrt{2}x \\ R\cosh\sqrt{5}x + S\sinh\sqrt{5}x \end{pmatrix}$$

Because

$$\begin{aligned} &3k_{11}e^{j\sqrt{2}x} + 3k_{12}e^{-j\sqrt{2}x} \\ &\quad = 3k_{11}\left(\cos\sqrt{2}x + j\sin\sqrt{2}x\right) + 3k_{12}\left(\cos\sqrt{2}x - j\sin\sqrt{2}x\right) \\ &\quad = P\cos\sqrt{2}x + Q\sin\sqrt{2}x \end{aligned}$$

where $P = 3k_{11} + 3k_{12}$ and $Q = (3k_{11} - 3k_{12})j$

and

$$\begin{aligned} &k_{21}e^{\sqrt{5}x} + k_{22}e^{-\sqrt{5}x} \\ &\quad = k_{21}\left(\cosh\sqrt{5}x + \sinh\sqrt{5}x\right) + k_{22}\left(\cosh\sqrt{5}x - \sinh\sqrt{5}x\right) \\ &\quad = R\cosh\sqrt{5}x + S\sinh\sqrt{5}x \text{ where } R = k_{21} + k_{22} \text{ and } S = k_{21} - k_{22} \end{aligned}$$

Therefore

$$\mathbf{F}(x) = \begin{pmatrix} \dots \\ \dots \end{pmatrix}$$

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$$\boxed{\mathbf{F}(x) = \begin{pmatrix} 3P\cos\sqrt{2}x + 3Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x \\ -4P\cos\sqrt{2}x - 4Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x \end{pmatrix}}$$

That is

$$f_1(x) = \ldots\ldots\ldots\ldots$$
$$f_2(x) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{aligned} f_1(x) &= 3P\cos\sqrt{2}x + 3Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x \\ f_2(x) &= -4P\cos\sqrt{2}x - 4Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x \end{aligned}}$$

Because

$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

and so

$$f_1(x) = 3P\cos\sqrt{2}x + 3Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x$$
$$f_2(x) = -4P\cos\sqrt{2}x - 4Q\sin\sqrt{2}x + R\cosh\sqrt{5}x + S\sinh\sqrt{5}x$$

Applying the boundary conditions, we find

$$\mathbf{F}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \ldots P + \ldots R \\ \ldots P + \ldots R \end{pmatrix} \text{ and } \mathbf{F}'(0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \ldots Q + \ldots S \\ \ldots Q + \ldots S \end{pmatrix}$$

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$$\boxed{\mathbf{F}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3P + R \\ -4P + R \end{pmatrix} \text{ and } \mathbf{F}'(0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3\sqrt{2}Q + \sqrt{5}S \\ -4\sqrt{2}Q + \sqrt{5}S \end{pmatrix}}$$

Because

$f_1(0) = 2$, $f_2(0) = 1$, $f_1'(0) = 4$ and $f_2'(0) = 3$ and so

$$\mathbf{F}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 3P + R \\ -4P + R \end{pmatrix} \text{ and}$$

$$\mathbf{F}'(0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} f_1'(0) \\ f_2'(0) \end{pmatrix} = \begin{pmatrix} 3\sqrt{2}Q + \sqrt{5}S \\ -4\sqrt{2}Q + \sqrt{5}S \end{pmatrix}$$

This gives the two sets of simultaneous equations

$$\begin{aligned} 3P + R &= 2 \\ -4P + R &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} 3\sqrt{2}Q + \sqrt{5}S &= 4 \\ -4\sqrt{2}Q + \sqrt{5}S &= 3 \end{aligned} \quad \text{which have solution}$$

$$P = \ldots\ldots\ldots\ldots, \quad R = \ldots\ldots\ldots\ldots,$$
$$Q = \ldots\ldots\ldots\ldots \text{ and } S = \ldots\ldots\ldots\ldots$$

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$$\boxed{P = 1/7,\ R = 11/7,\ Q = 1/\left(7\sqrt{2}\right) \text{ and } S = 25/\left(7\sqrt{5}\right)}$$

This gives the complete solution as

$$f_1(x) = \frac{3}{7}\cos\sqrt{2}x + \frac{3}{7\sqrt{2}}\sin\sqrt{2}x + \frac{11}{7}\cosh\sqrt{5}x + \frac{25}{7\sqrt{5}}\sinh\sqrt{5}x$$
$$f_2(x) = -\frac{4}{7}\cos\sqrt{2}x - \frac{4}{7\sqrt{2}}\sin\sqrt{2}x + \frac{11}{7}\cosh\sqrt{5}x + \frac{25}{7\sqrt{5}}\sinh\sqrt{5}x$$

This method is quite straightforwardly extended to three or more such coupled differential equations.

Summary

To solve the system of coupled second-order differential equations

$$\mathbf{F}''(x) = \mathbf{AF}(x)$$

1. Find the eigenvalues and eigenvectors of matrix **A** and construct the modal matrix **M** and the diagonal spectral matrix **S**
2. Solve the equation $\mathbf{G}'(x) = \mathbf{SG}(x)$ (note that even though $\mathbf{M}^{-1}$ is used there was no need to calculate it)
3. Apply $\mathbf{F}(x) = \mathbf{MG}(x)$ to find $\mathbf{F}(x)$.

Try one yourself.

Next frame

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Example 2

The system of coupled second-order differential equations (refer to Frames 57 to 61)

$$f_1''(x) = 3f_1(x) + 10f_2(x)$$
$$f_2''(x) = 2f_1(x) + 4f_2(x)$$
where $f_1(0) = 0$, $f_2(0) = 1$, $f_1'(0) = 1$ and $f_2'(0) = 0$

has the solution (refer to Frames 57 to 61)

$$f_1(x) = \ldots\ldots\ldots\ldots$$
$$f_2(x) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{aligned} f_1(x) &= 10\cos x + \frac{5}{9}\sin x + 10\cosh 2\sqrt{2}x + \frac{2}{9\sqrt{2}}\sinh 2\sqrt{2}x \\ f_2(x) &= -4\cos x - \frac{2}{9}\sin x + 5\cosh 2\sqrt{2}x + \frac{1}{9\sqrt{2}}\sinh 2\sqrt{2}x \end{aligned}}$$

Because

$$f_1''(x) = 3f_1(x) + 10f_2(x)$$
$$f_2''(x) = 2f_1(x) + 4f_2(x)$$

can be written in matrix form as

....

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$$\boxed{\begin{pmatrix} f_1''(x) \\ f_2''(x) \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}}$$

That is

$$\mathbf{F}''(x) = \mathbf{A}\mathbf{F}(x)$$

where $\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $\mathbf{F}''(x) = \begin{pmatrix} f_1''(x) \\ f_2''(x) \end{pmatrix}$ and $\mathbf{A} = \begin{pmatrix} 3 & 10 \\ 2 & 4 \end{pmatrix}$

and where $\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{F}'(0) = \begin{pmatrix} f_1'(0) \\ f_2'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

To solve the matrix equation we first need the eigenvalues and associated eigenvectors of the matrix $\mathbf{A}$. These have already been found in Frames 57 to 61 and they are

$\lambda = -1$ with corresponding eigenvector $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$\lambda = 8$ with corresponding eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The complete solution of $\mathbf{F}'' = \mathbf{AF}$ is then

$$\begin{aligned} \mathbf{F}(x) &= (P\cos x + Q\sin x)\begin{pmatrix} 5 \\ -2 \end{pmatrix} + \left(R\cosh 2\sqrt{2}x + S\sinh 2\sqrt{2}x\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 5P\cos x + 5Q\sin x + 2R\cosh 2\sqrt{2}x + 2S\sinh 2\sqrt{2}x \\ -2P\cos x - 2Q\sin x + R\cosh 2\sqrt{2}x + S\sinh 2\sqrt{2}x \end{pmatrix} \end{aligned}$$

That is

$$f_1(x) = \ldots\ldots\ldots\ldots$$
$$f_2(x) = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{aligned} f_1(x) &= 5P\cos x + 5Q\sin x + 2R\cosh 2\sqrt{2}x + 2S\sinh 2\sqrt{2}x \\ f_2(x) &= -2P\cos x - 2Q\sin x + R\cosh 2\sqrt{2}x + S\sinh 2\sqrt{2}x \end{aligned}}$$

Because

$$\mathbf{F}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

and so

$$f_1(x) = 5P\cos x + 5Q\sin x + 2R\cosh 2\sqrt{2}x + 2S\sinh 2\sqrt{2}x$$
$$f_2(x) = -2P\cos x - 2Q\sin x + R\cosh 2\sqrt{2}x + S\sinh 2\sqrt{2}x$$

Applying the boundary conditions, we find

$$\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \ldots P + \ldots R \\ \ldots P + \ldots R \end{pmatrix} \text{ and } \mathbf{F}'(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \ldots Q + \ldots S \\ \ldots Q + \ldots S \end{pmatrix}$$

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$$\boxed{\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5P + 2R \\ -2P + R \end{pmatrix} \text{ and } \mathbf{F}'(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5Q + 4\sqrt{2}S \\ -2Q + 2\sqrt{2}S \end{pmatrix}}$$

Because

The boundary conditions are $f_1(0) = 0$, $f_2(0) = 1$, $f_1'(0) = 1$ and $f_2'(0) = 0$, therefore

$$\mathbf{F}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 5P + 2R \\ -2P + R \end{pmatrix} \text{ and}$$
$$\mathbf{F}'(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} f_1'(0) \\ f_2'(0) \end{pmatrix} = \begin{pmatrix} 5Q + 4\sqrt{2}S \\ -2Q + 2\sqrt{2}S \end{pmatrix}$$

This gives the two sets of simultaneous equations

$$\begin{aligned} 5P + 2R &= 0 \\ -2P + R &= 1 \end{aligned} \quad \text{and} \quad \begin{aligned} 5Q + 4\sqrt{2}S &= 1 \\ -2Q + 2\sqrt{2}S &= 0 \end{aligned} \quad \text{which have solution}$$

$$P = \ldots\ldots\ldots\ldots,\quad R = \ldots\ldots\ldots\ldots,$$
$$Q = \ldots\ldots\ldots\ldots \text{ and } S = \ldots\ldots\ldots\ldots$$

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$$\boxed{P = -2/9,\quad R = 5/9,\quad Q = 1/9 \text{ and } S = 1/\left(9\sqrt{2}\right)}$$

This gives the complete solution as

$$f_1(x) = -\frac{10}{9}\cos x + \frac{5}{9}\sin x + \frac{10}{9}\cosh 2\sqrt{2}x + \frac{2}{9\sqrt{2}}\sinh 2\sqrt{2}x$$
$$f_2(x) = \frac{4}{9}\cos x - \frac{2}{9}\sin x + \frac{5}{9}\cosh 2\sqrt{2}x + \frac{1}{9\sqrt{2}}\sinh 2\sqrt{2}x$$

Matrix transformation

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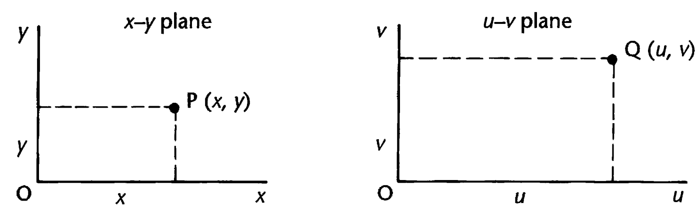


If for every point Q (u, v) in the u–v plane there is a corresponding point P (x, y) in the x–y plane, then there is a relationship between the two sets of coordinates. In the simple case of scaling the coordinate where

$$u = ax \text{ and } v = by$$

we have a *linear transformation* and we can combine these in matrix form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ then provides the transformation between the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in one set of coordinates and the vector $\begin{pmatrix} u \\ v \end{pmatrix}$ in the other set of coordinates.

Similarly, if we solve the two equations for x and y, we have

$$x = \frac{1}{a}u \quad \text{and} \quad y = \frac{1}{b}v$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

which allows us to transform back from the u–v plane coordinates to the x–y plane coordinates.

Now for an example.

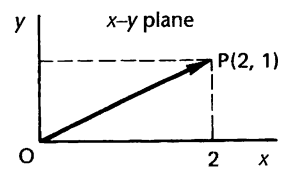
▶

Example

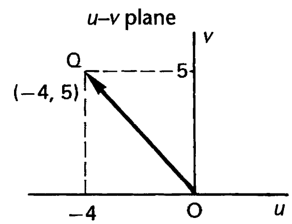
If $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ with the transformation $\mathbf{T} = \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix}$ determine $\mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{TX}$ and show the positions on the x–y and u–v planes.

In this case

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$



transforms into



If $\mathbf{T}$ is non-singular and $\mathbf{U} = \mathbf{TX}$ then $\mathbf{X} = \mathbf{T}^{-1}\mathbf{U}$ and since

$$\mathbf{T} = \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix} \text{ then } \mathbf{T}^{-1} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\mathbf{T}^{-1} = \begin{pmatrix} -1/2 & 0 \\ 1 & 1 \end{pmatrix}}$$

There are several ways of finding the inverse of a matrix. One method is as follows.

$$\mathbf{T} = \begin{pmatrix} -2 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} -2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

$$\therefore\ \mathbf{T}^{-1} = \begin{pmatrix} -1/2 & 0 \\ 1 & 1 \end{pmatrix}$$

So we have $\mathbf{U} = \mathbf{TX}\quad \therefore\ \mathbf{X} = \mathbf{T}^{-1}\mathbf{U}$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

Hence a vector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ in the u–v plane transforms into $\begin{pmatrix} x \\ y \end{pmatrix}$ in the x–y plane where $\begin{pmatrix} x \\ y \end{pmatrix} = \ldots\ldots\ldots\ldots$

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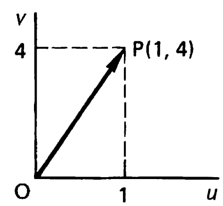
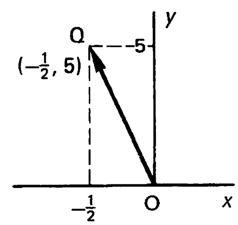
$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2 \\ 5 \end{pmatrix}}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 5 \end{pmatrix}$$

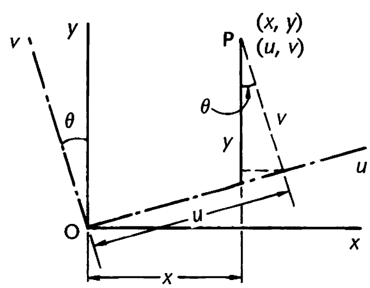


transforms into



Rotation of axes

A more interesting case occurs with a degree of rotation between the two sets of coordinate axes.



Let P be the point (x, y) in the x–y plane and the point (u, v) in the u–v plane.

Let θ be the angle of rotation between the two systems. From the diagram we can see that

$$\left.\begin{aligned} x &= u\cos\theta - v\sin\theta \\ y &= u\sin\theta + v\cos\theta \end{aligned}\right\} \qquad (1)$$

In matrix form, this becomes $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

which enables us to transform from the u–v plane coordinates to the corresponding x–y plane coordinates.

Make a note of this and then move on

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If we solve equations (1) for u and v, we have

$$x\sin\theta = u\sin\theta\cos\theta - v\sin^2\theta$$
$$y\cos\theta = u\sin\theta\cos\theta + v\cos^2\theta$$
$$\therefore\ y\cos\theta - x\sin\theta = v(\cos^2\theta + \sin^2\theta) = v$$

Also
$$x\cos\theta = u\cos^2\theta - v\sin\theta\cos\theta$$
$$y\sin\theta = u\sin^2\theta + v\sin\theta\cos\theta$$
$$\therefore\ x\cos\theta + y\sin\theta = u(\cos^2\theta + \sin^2\theta) = u$$

So
$$u = x\cos\theta + y\sin\theta$$
$$v = -x\sin\theta + y\cos\theta$$

and written in matrix form, this is ....

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$$\boxed{\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}$$

So we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

and
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

i.e. $\mathbf{X} = \mathbf{TU}$ and $\mathbf{U} = \mathbf{T}^{-1}\mathbf{X}$

where $\mathbf{T}$ is the matrix of transformation and the equations provide a linear transformation between the two sets of coordinates.

Example

If the u–v plane axes rotate through 30° in an anticlockwise manner from the x–y plane axes, determine the (u, v) coordinates of a point whose (x, y) coordinates are $x = 2$, $y = 3$ in the x–y plane.

This is a straightforward application of the results above.

So $\begin{pmatrix} u \\ v \end{pmatrix} = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \sqrt{3} + 3/2 \\ -1 + 3\sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 3{\cdot}23 \\ 1{\cdot}60 \end{pmatrix}}$$

Because

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \begin{array}{l} \cos\theta = \sqrt{3}/2 \\ \sin\theta = 1/2 \end{array} \\ &= \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{3} + 3/2 \\ -1 + 3\sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 3{\cdot}23 \\ 1{\cdot}60 \end{pmatrix} \end{aligned}$$

As usual, the Programme ends with the **Revision summary**, to be read in conjunction with the **Can You?** checklist. Go back to the relevant part of the Programme for any points on which you are unsure. The **Test exercise** should then be straightforward and the **Further problems** give valuable additional practice.

Revision summary 12

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1 *Singular* square matrix: $|\mathbf{A}| = 0$

Non-singular square matrix $|\mathbf{A}| \neq 0$.

2 *Rank of a matrix* – order of the largest non-zero determinant that can be formed from the elements of the matrix.

3 *Elementary operations and equivalent matrices*
Each of the following row operations on matrix **A** produces a *row equivalent matrix* **B** where the order and rank of **B** are the same as those of **A**. We write $\mathbf{A} \sim \mathbf{B}$.

(1) Interchanging two rows
(2) Multiplying each element of a row by the same non-zero scalar quantity
(3) Adding or subtracting corresponding elements from those of another row.

These operations are called *elementary row operations*. There is a corresponding set of three *elementary column operations* that can be used to form *column equivalent matrices*.

4 *Consistency* of a set of n equations in n unknowns with coefficient matrix **A** and augmented matrix $\mathbf{A}_b$.

(a) Consistent if rank of $\mathbf{A}$ = rank of $\mathbf{A}_b$
(b) Inconsistent if rank of $\mathbf{A}$ < rank of $\mathbf{A}_b$.

▶

5 *Uniqueness of solutions – n equations with n unknowns.*

(a) rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = n$ — *unique solutions*

(b) rank of $\mathbf{A}$ = rank of $\mathbf{A}_b = m < n$ — *infinite number of solutions*

(c) rank of $\mathbf{A} <$ rank of $\mathbf{A}_b$ — *no solution*

6 *Solution of sets of equations*

(a) *Inverse matrix method* $\mathbf{Ax} = \mathbf{b}$; $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

To find $\mathbf{A}^{-1}$

(1) evaluate $|\mathbf{A}|$

(2) form $\mathbf{C}$, the matrix of cofactors of $\mathbf{A}$

(3) write $\mathbf{C}^{\mathrm{T}}$, the transpose of $\mathbf{A}$

(4) $\mathbf{A}^{-1} = \dfrac{1}{|\mathbf{A}|} \times \mathbf{C}^{\mathrm{T}}$.

(b) *Row transformation method* $\mathbf{Ax} = \mathbf{b}$; $\mathbf{Ax} = \mathbf{Ib}$

(1) form the combined coefficient matrix $[\mathbf{A} \mid \mathbf{I}]$

(2) row transformations to convert to $[\mathbf{I} \mid \mathbf{A}^{-1}]$

(3) then solve $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

(c) *Gaussian elimination method* $\mathbf{Ax} = \mathbf{b}$

(1) form augmented matrix $[\mathbf{A} \mid \mathbf{b}]$

(2) operate on rows to convert to $[\mathbf{U} \mid \mathbf{b}']$ where $\mathbf{U}$ is the upper-triangular matrix.

(3) expand from bottom row to obtain $\mathbf{x}$.

(d) *Triangular decomposition method* $\mathbf{Ax} = \mathbf{b}$

Write $\mathbf{A}$ as the product of upper and lower triangular matrices.

$\mathbf{A} = \mathbf{LU}$, $\mathbf{L}(\mathbf{Ux}) = \mathbf{b}$. Put $\mathbf{Ux} = \mathbf{y}$ $\therefore$ $\mathbf{Ly} = \mathbf{b}$

(1) solve $\mathbf{Ly} = \mathbf{b}$ to obtain $\mathbf{y}$

(2) solve $\mathbf{Ux} = \mathbf{y}$ to obtain $\mathbf{x}$.

7 *Eigenvalues and eigenvectors* $\mathbf{Ax} = \lambda\mathbf{x}$

Sets of equations of form $\mathbf{Ax} = \lambda\mathbf{x}$, where $\mathbf{A}$ = coefficient matrix, $\mathbf{x}$ = column matrix, λ = scalar quantity.

Equations become $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$.

For non-trivial solutions, $|\mathbf{A} - \lambda\mathbf{I}| = 0$ is the *characteristic equation* and gives values of λ i.e. the *eigenvalues*.

Substitution of each eigenvalue gives a corresponding *eigenvector*.

8 *Cayley–Hamilton theorem*

Every square matrix satisfies its own characteristic equation.

9 *Solving systems of first-order ordinary differential equations*

To solve the system of coupled first-order differential equations

$$\mathbf{F}'(x) = \mathbf{AF}(x)$$

▶

(a) Find the eigenvalues and eigenvectors of matrix **A** and construct the modal matrix **M** and the diagonal spectral matrix **S**

(b) Solve the equation $\mathbf{G}'(x) = \mathbf{S}\mathbf{G}(x)$

(c) Apply $\mathbf{F}(x) = \mathbf{M}\mathbf{G}(x)$ to find $\mathbf{F}(x)$.

10 *Diagonalisation of a matrix*

Modal matrix of **A**

If **A** has distinct eigenvalues $\mathbf{M} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$, where $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_n$ are eigenvectors of **A**, then $\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \mathbf{S}$ where **S** is the *spectral matrix* of **A**

$$\text{and} \quad \mathbf{S} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

$\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of **A**.

11 *Solving systems of second-order ordinary differential equations*

To solve an equation of the form

$$\mathbf{F}''(x) = \mathbf{A}\mathbf{F}(x)$$

(a) Find the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of **A**

(b) Assuming the eigenvectors are all distinct, find the associated eigenvectors $\mathbf{C}_1, \mathbf{C}_2, \dots, \mathbf{C}_n$

(c) Write the solution of the equation as

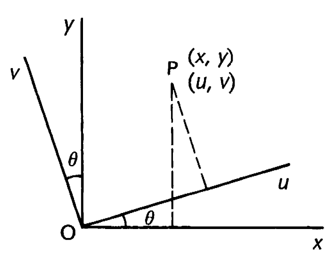
$$\mathbf{F}(x) = \sum_{r=1}^{n} \left(a_r e^{\sqrt{\lambda_r}x} + b_r e^{-\sqrt{\lambda_r}x} \right) \mathbf{C}_r$$

and use the boundary conditions to find the values of a_r and b_r for $r = 1, 2, \dots, n$.

12 *Matrix transformation*

(a) $\mathbf{U} = \mathbf{T}\mathbf{X}$, where **T** is a transformation matrix, transforms a vector in the x–y plane to a corresponding vector in the u–v plane. Similarly, $\mathbf{X} = \mathbf{T}^{-1}\mathbf{U}$ converts a vector in the u–v plane to a corresponding vector in the x–y plane.

(b) *Rotation of axes*



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Can You?

110 Checklist 12

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Determine whether a matrix is singular or non-singular? 1 to 3

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine the rank of a matrix? 3 to 13

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Determine the consistency of a set of linear equations and hence demonstrate the uniqueness of their solution? 14 to 23

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the solution of a set of simultaneous linear equations by using matrix inversion, by row transformation, by Gaussian elimination and by triangular decomposition? 24 to 52

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the eigenvalues and corresponding eigenvectors of a square matrix? 53 to 65

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Demonstrate the validity of the Cayley–Hamilton theorem? 66 and 67

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve systems of first-order ordinary differential equations using eigenvalue and eigenvector methods? 68 to 79

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Construct the modal matrix from the eigenvectors of a matrix and the spectral matrix from the eigenvalues? 80 to 86

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve systems of second-order ordinary differential equations using diagonalisation? 87 to 102

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use matrices to represent transformations between coordinate systems? 103 to 108

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 12

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1 Determine the rank of **A** and of $\mathbf{A}_b$ for the following sets of equations and hence determine the nature of the solutions. Do *not* solve the equations.

(a) $$\begin{aligned} x_1 + 3x_2 - 2x_3 &= 6 \\ 4x_1 + 5x_2 + 2x_3 &= 3 \\ x_1 + 3x_2 + 4x_3 &= 7 \end{aligned}$$

(b) $$\begin{aligned} x_1 + 2x_2 - 4x_3 &= 3 \\ x_1 + 2x_2 + 3x_3 &= -4 \\ 2x_1 + 4x_2 + x_3 &= -3. \end{aligned}$$

2 If $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 2 & 3 & -2 \\ 3 & 5 & -4 \\ 1 & 2 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 10 \\ 9 \end{pmatrix}$, determine $\mathbf{A}^{-1}$ and hence solve the set of equations.

3 Given that
$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_1 - x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 - 2x_3 &= 0 \end{aligned}$$
apply the method of row transformation to obtain the value of x_1, x_2, x_3.

4 By the method of Gaussian elimination, solve the equations $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \begin{pmatrix} 1 & -2 & -4 \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$.

5 If $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 5 & 3 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$, express **A** as the product $\mathbf{A} = \mathbf{LU}$ where **L** and **U** are lower and upper-triangular matrices and hence determine the values of x_1, x_2, x_3.

6 Determine the eigenvalues and corresponding eigenvectors of $\mathbf{Ax} = \lambda\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$.

7 If $\mathbf{x_1}$ and $\mathbf{x_2}$ are eigenvectors of $\mathbf{Ax} = \lambda\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$ determine

(a) $\mathbf{M} = (\mathbf{x}_1\mathbf{x}_2)$

(b) $\mathbf{M}^{-1}$

(c) $\mathbf{M}^{-1}\mathbf{AM}$.

8 Solve the system of first-order differential equations
$$\begin{aligned} f_1'(x) &= 5f_1(x) - 2f_2(x) \\ f_2'(x) &= -f_1(x) + 4f_2(x) \end{aligned}$$
where $f_1(0) = -3$ and $f_2(0) = 2$.

9 Solve the system of second-order differential equations

$$\begin{aligned} f_1''(x) &= f_1(x) + 6f_2(x) \\ f_2''(x) &= 3f_1(x) - 2f_2(x) \end{aligned} \quad \text{where } f_1(0) = 1,\ f_2(0) = 0,\ f_1'(0) = 2,\ f_2'(0) = -1.$$

10 (a) Determine the vector in the u–v plane formed by $\mathbf{U} = \mathbf{TX}$, where the transformation matrix is $\mathbf{T} = \begin{pmatrix} -2 & 1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is a vector in the x–y plane.

(b) The coordinate axes in the x–y plane and in the u–v plane have the same origin O, but OU is inclined to OX at an angle of 60° in an anticlockwise manner. Transform a vector $\mathbf{X} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$ in the x–y plane into the corresponding vector in the u–v plane.

Further problems 12

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1 If $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 5 & 2 & 3 \\ 3 & -2 & -2 \\ 4 & 3 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ 5 \\ -5 \end{pmatrix}$, determine $\mathbf{A}^{-1}$ and hence solve the set of equations.

2 Apply the method of row transformation to solve the following sets of equations.

(a)
$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 8 \\ 2x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 - 4x_2 + 2x_3 &= -3 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= 8 \\ 2x_1 - x_2 + x_3 &= 9 \\ 3x_1 + 2x_2 + 3x_3 &= 5. \end{aligned}$$

3 Solve the following sets of equations by Gaussian elimination.

(a)
$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 4 \\ 2x_1 + x_2 + x_3 - 4x_4 &= 3 \\ 3x_1 - x_2 - 2x_3 + 2x_4 &= 6 \\ x_1 + 3x_2 - x_3 + x_4 &= 8 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 + 2x_4 &= 2 \\ 4x_1 + 2x_2 - 3x_3 - x_4 &= 6 \\ x_1 - x_2 + 4x_3 - 2x_4 &= 7 \\ 3x_1 + 2x_2 + x_3 - x_4 &= 5 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 + 5x_3 + x_4 &= 4 \\ 3x_1 - 4x_2 + 3x_3 - 2x_4 &= 7 \\ 4x_1 + 3x_2 + 2x_3 - x_4 &= 1 \\ x_1 - 2x_2 - 4x_3 - x_4 &= 2. \end{aligned}$$

4 Using the method of triangular decomposition, solve the following sets of equations.

(a) $$\begin{pmatrix} 1 & 4 & -1 \\ 4 & 2 & 3 \\ 7 & -3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -18 \end{pmatrix}$$

(b) $$\begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \\ 6 & -3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 17 \\ 22 \end{pmatrix}$$

(c) $$\begin{pmatrix} 1 & -2 & 3 & -1 \\ 3 & 1 & -3 & 2 \\ 5 & 3 & 2 & 3 \\ 2 & -4 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ 14 \\ 21 \\ -10 \end{pmatrix}.$$

5 If $\mathbf{Ax} = \lambda\mathbf{x}$, determine the eigenvalues and corresponding eigenvectors in each of the following cases.

(a) $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & -4 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} -6 & 5 \\ 4 & 2 \end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix} -5 & 9 \\ 1 & 3 \end{pmatrix}$

(e) $\mathbf{A} = \begin{pmatrix} 2 & 7 & 0 \\ 1 & 3 & 1 \\ 5 & 0 & 8 \end{pmatrix}$ (f) $\mathbf{A} = \begin{pmatrix} 5 & -6 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$

(g) $\mathbf{A} = \begin{pmatrix} -3 & 0 & 6 \\ 4 & 5 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ (h) $\mathbf{A} = \begin{pmatrix} 4 & 10 & -8 \\ 1 & 2 & 1 \\ -1 & 2 & 3 \end{pmatrix}.$

6 Solve each of the following systems of first-order differential equations.

(a) $f_1'(x) = 2f_1(x) - 5f_2(x)$
$f_2'(x) = f_1(x) - 4f_2(x)$
where $f_1(0) = 1$ and $f_2(0) = 0$

(b) $f_1'(x) = -5f_1(x) + 9f_2(x)$
$f_2'(x) = f_1(x) + 3f_2(x)$
where $f_1(0) = 0$ and $f_2(0) = -2$

(c) $f_1'(x) = 5f_1(x) - 6f_2(x) + f_3(x)$
$f_2'(x) = f_1(x) + f_2(x)$
$f_3'(x) = 3f_1(x) + f_3(x)$
where $f_1(0) = 1$, $f_2(0) = 0$ and $f_3(0) = 2$

(d) $f_1'(x) = 4f_1(x) + 10f_2(x) - 8f_3(x)$
$f_2'(x) = f_1(x) + 2f_2(x) + f_3(x)$
$f_3'(x) = -f_1(x) + 2f_2(x) + 3f_3(x)$
where $f_1(0) = 4$, $f_2(0) = -2$ and $f_3(0) = -1$.

▶

7 If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 10 & -3 \\ 0 & -3 & 9 \end{pmatrix}$, determine the three eigenvalues λ_1, λ_2, λ_3 of $\mathbf{A}$

and verify that if $\mathbf{M} = \begin{pmatrix} -9 & 1 & 1 \\ 3 & 2 & 4 \\ 1 & 3 & -3 \end{pmatrix}$ then $\mathbf{M}^{-1}\mathbf{AM} = \mathbf{S}$, where $\mathbf{S}$ is a diagonal matrix with elements λ_1, λ_2, λ_3.

8 Invert the matrix $\mathbf{A} = \begin{pmatrix} 8 & 10 & 7 \\ 5 & 9 & 4 \\ 9 & 11 & 8 \end{pmatrix}$ and hence solve the equations

$$\begin{aligned} 8I_1 + 10I_2 + 7I_3 &= 0 \\ 5I_1 + 9I_2 + 4I_3 &= -9 \\ 9I_1 + 11I_2 + 8I_3 &= 1. \end{aligned}$$

9 If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 5 & 8 & 9 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 6 & -4 \\ -1 & -6 & 5 \\ 2 & 2 & -2 \end{pmatrix}$, verify that $\mathbf{AB} = k\mathbf{I}$ where $\mathbf{I}$ is a unit matrix and k is a constant. Hence solve the equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2 \\ 4x_1 + 6x_2 + 7x_3 &= 2 \\ 5x_1 + 8x_2 + 9x_3 &= 3. \end{aligned}$$

10 Solve each of the following systems of second-order differential equations.

(a) $f_1''(x) = 4f_1(x) + 3f_2(x)$
$f_2''(x) = 2f_1(x) + 5f_2(x)$
where $f_1(0) = 0$, $f_2(0) = 1$, $f_1'(0) = 4$ and $f_2'(0) = 1$

(b) $f_1''(x) = -6f_1(x) + 5f_2(x)$
$f_2''(x) = 4f_1(x) + 2f_2(x)$
where $f_1(0) = 0$, $f_2(0) = 1$, $f_1'(0) = 1$ and $f_2'(0) = 0$

(c) $f_1''(x) = 2f_1(x) + 7f_2(x)$
$f_2''(x) = f_1(x) + 3f_2(x) + f_3(x)$
$f_3''(x) = 5f_1(x) + 8f_3(x)$
where $f_1(0) = 1$, $f_2(0) = 1$, $f_3(0) = 0$, $f_1'(0) = 0$, $f_2'(0) = 0$ and $f_3'(0) = 1$

(d) $f_1''(x) = -3f_1(x) + 6f_3(x)$
$f_2''(x) = 4f_1(x) + 5f_2(x) + 3f_3(x)$
$f_3''(x) = f_1(x) + 2f_2(x) + f_3(x)$
where $f_1(0) = 1$, $f_2(0) = 1$, $f_3(0) = 0$, $f_1'(0) = 0$, $f_2'(0) = 0$, $f_3'(0) = 1$.